LINEAR ALGEBRA RESEARCH PROJECT 2014

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1. Set up

To set up this project will require some working knowledge of complex numbers and calculus. Let's begin with notations and definitions. Let $\mathbb{R}, \mathbb{C}$ respectively denote the sets of real numbers and complex numbers. For $z \in \mathbb{C}$, we shall write

$$z = x + iy$$

where $i = \sqrt{-1}$ (hence $i^2 = -1$), and $x, y \in \mathbb{R}$, which are respectively called the *real* and *imaginary* parts of z. The *complex conjugate* of z is then

$$\bar{z} := x - iy$$

and its *absolute value* is

$$|z| := \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Using these definitions, we can treat the set $\mathbb{C}$ and the more familiar Euclidean plane $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$ as one and the same thing, without losing any information. In other words, we can *identify* the two objects under the correspondence

$$\mathbb{C} \ni z = x + iy \leftrightarrow (x, y) \in \mathbb{R}^2.$$

Note that under this identification, $|z|$ is exactly the familiar Euclidean length of the vector $(x, y) \in \mathbb{R}^2$.

Let $c_0, .., c_k \in \mathbb{C}$. A *polynomial* in one variable z of *degree* $k \geq 0$ is a function of the form

$$p(z) = c_0 + c_1 z + \cdots + c_k z^k$$

where $c_0, .., c_k \in \mathbb{C}$; c_j is called the *jth coefficient* of $p(z)$. Let $\mathbb{C}[z]$ denote the set of all polynomials in z. A *rational function* $f(z)$ is function of the form

$$f(z) = p(z)/q(z)$$

where $p(z), q(z) \in \mathbb{C}[z]$ with $q(z) \neq 0$.

As in calculus, we can define the first derivative of the polynomial $p(z)$ above to be the polynomial

$$p'(z) := \frac{d}{dz} p(z) := c_1 + 2c_2 z + \cdots + kc_k z^{k-1}.$$

The second, third, and higher derivatives $p''(z), p'''(z), ..., p^{(s)}(z)$ of $p(z)$ can be defined similarly. Likewise, for the rational function $f(z)$ above, we have

$$f'(z) = \frac{q(z)p'(z) - p(z)q'(z)}{q(z)^2}.$$

It is not hard to verify that the usual rules of differentiation such as the chain rule and the Leibniz rule hold for polynomials and rational functions.

Let us write $c_j = a_j + ib_j$ where $a_j, b_j \in \mathbb{R}$ are the ***real*** and ***imaginary*** parts of c_j; likewise for $z = x + iy$ as before. Then by expanding each power $z^j = (x+iy)^j$ appearing in $p(z)$ above (using ($i^2 = -1$), we see that

$$p(z) = P(x,y) + iQ(x,y)$$

where $P(x,y)$ is a real-valued function of the form $\sum_{0\leq m,n\leq k} C_{m,n}x^m y^n$ where the $C_{m,n} \in \mathbb{R}$ can be simply expressed in terms the a_j, b_j. Likewise for the function $Q(x,y)$; P, Q are respectively called the real and imaginary parts of $p(z)$. For example, in the degree 1 case

$$p(z) = (a_0 + ib_0) + (a_1 + ib_1)(x+iy) = (a_0 + a_1 x - b_1 y) + i(b_0 + b_1 x + a_1 y).$$

In the degree 2 case,

$$p(z) = (a_0 + a_1 x - b_1 y + a_2 x^2 - a_2 y^2 - 2b_2 xy) + i(b_0 + b_1 x + a_1 y + b_2 x^2 - b_2 y^2 + 2a_2 xy).$$

Similarly, if $f(z) = p(z)/q(z)$ is a rational function we can also write

$$f(z) = A(x,y) + iB(x,y)$$

where $A(x,y), B(x,y)$ are real-valued functions expressible in terms of the real and imaginary parts of the polynomials $p(z), q(z)$.

2. Contour Integration of Rational Functions

Let C be a parameterized smooth curve in the complex plane $\mathbb{C}$. By this, we simply mean that we are given a $\mathbb{C}$-valued function defined on the interval $[0,1] = \{t \in \mathbb{R} | 0 \leq t \leq 1\}$:

$$[0,1] \to \mathbb{C},\ t \mapsto z(t) = x(t) + iy(t)$$

that is continuously differentiable and that

$$z'(t) \equiv \frac{dz}{dt} := \frac{dx}{dt} + i\frac{dy}{dt} \neq 0, \qquad t \in [0,1].$$

Intuitively, we can think of this as a way to "trace" the directed curve C in the complex plane $\mathbb{C}$ (or the Euclidean plane $\mathbb{R}^2$) from the point $z(0)$ to the point $z(1)$ without stopping, as time t goes from 0 to 1.

Now, let $f(z) = p(z)/q(z)$ be a rational function that is nonsingular along C, i.e. $q(z(t)) \neq 0$ for $t \in [0,1]$. We would like to give meaning to the "integral" of $f(z)$ along C:

$$\int_C f(z)dz.$$

Writing $f(z) = A(x,y) + iB(x,y)$ and $z(t) = x(t) + iy(t)$ as before, and by formal manipulation we would get

$$\int_C f(z)dz = \int_0^1 f(z(t))\frac{dz}{dt}dt \quad = \int_0^1 \left\{(A(x(t),y(t))\tfrac{dx}{dt} - B(x(t),y(t))\tfrac{dy}{dt}\right\}dt$$
$$+\, i\int_0^1 \left\{(A(x(t),y(t))\tfrac{dy}{dt} + B(x(t),y(t))\tfrac{dx}{dt}\right\}dt.$$

Since the right hand side now makes sense as a complex number, we can simply ***define*** $\int_C f(z)dz$ to mean just that. This number is called the ***contour integral*** of f along C. It is not hard to show that $\int_C f(z)dz$ defined this way, in fact, does not depend on the choice of the parameterization $z(t)$ of C. In other words, if we replace $z(t)$ with any other such parameterization $\tilde{z} = \tilde{x} + i\tilde{y} : [0,1] \to \mathbb{C}$ of the same directed curve C, then the right hand side integral above will remain unchanged.

For this project, it is not important to go into such generality because we will only be working with one specific curve, namely, a counterclockwise circle of radius r centered at 0. For this, we can just use the familiar parameterization:

$$z(t) = r\cos(2\pi t) + ir\sin(2\pi t), \qquad t \in [0,1].$$

From now on, C shall mean this circle. There is famous formula due to L. Euler that says that

$$e^{it} = \cos(t) + i\sin(t)$$

for any real number t. Therefore we have $z(t) = re^{2\pi i t}$.

We will consider a general problem about contour integrals, starting from a number of important special cases. A related second problem will be described in the next section. Students should feel free to work on any or all of these. Explorations of the first few should requires only some basic computational skills in one-variable calculus. Students should not think that one has to solve a problem in full in order for their work to be worthwhile. In fact, a genuine and persistent effort of trying is far more important that getting a complete answer to a problem. Do remember that problem-solving at the research level is not a competition – it is much more of a process of learning and enlightenment.

One useful tip. It is not always necessary to get a complete proof in every steps, in order to to gain real insights into any one of these problems. Very often, it is necessary and better to assume that certain statements that you believe are true (but you can't yet prove immediately) along the way, rather than getting bogged down by technical steps. If the assumptions you made lead you to new understanding of the problem, or even a solution to the problem, you have then reduced the problem to just checking those assumptions that you have made. And that usually represents real progress!

Good luck to you all!

Problem 1. *Show that $\int_C z^k dz$ is $2\pi i$ if $k = -1$, and is 0 otherwise.*

For the next problem, you can try the case when $a \in \mathbb{R}$ and assume

$$\frac{d}{da}\int_C \frac{dz}{z-a} = \int_C \frac{d}{da}\frac{dz}{z-a}$$

to try to get some insights first.

Problem 2. *For $a \in \mathbb{C}$, argue that $\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = \begin{cases} 0 & \text{if } |a| > r \\ 1 & \text{if } |a| < r \end{cases}$.*

Problem 3. *Prove the "integration by parts" formula:*

$$\int_C \frac{d}{dz} f(z)dz = 0$$

for any rational function $f(z)$.

Problem 4. *Show that for $p(z) \in \mathbb{C}[z]$, $|a| < r$ and $n \geq 0$,*

$$\frac{1}{2\pi i}\int_C \frac{p(z)dz}{(z-a)^{n+1}} = \frac{1}{n!}p^{(n)}(a).$$

Problem 5. *For $a, b \in \mathbb{C}$ distinct and enclosed by the circle C, compute*

$$\frac{1}{2\pi i}\int_C \frac{dz}{(z-a)(z-b)}.$$

3. General Problem

Problem 6. *Let $f(z) = p(z)/q(z)$ be a rational function, and C be the counterclockwise circle of radius r enclosing all zeros of the polynomial $q(z)$, i.e. if $q(a) = 0$ then $|a| < r$. Compute the contour integral $\frac{1}{2\pi i}\int_C f(z)dz$ explicitly, i.e. find a closed formula for this integral in terms of the given data p, q.*

To make this problem a bit more concrete, let us reduce it further using the famous **Fundamental Theorem of Algebra:** *Every non-constant polynomial in $\mathbb{C}[z]$ has a zero.*

Then by using the division algorithm, it is not hard to show that any given non-constant polynomial $q(z)$ can be uniquely factorized into the form

$$q(z) = b(z-a_1)^{n_1+1}\cdots(z-a_m)^{n_m+1}$$

where $b, a_1, .., a_m \in \mathbb{C}$ and the n_j are nonnegative integers. This means that we can specify every given polynomial $q(z)$ by specifying the data $b, a_1, .., a_m$ and $n_1, .., n_m$ (which together give more information than just giving $q(z)$ itself!) Thus, the problem becomes one of computing $\frac{1}{2\pi i}\int_C f(z)dz$ in terms of the data $b, a_1, .., a_m$, $n_1, .., n_m$, and the polynomial $p(z) = \sum_{j=0}^{k} c_j z^j$. Note that Problem 2.3 is the special case with $q(z) = (z-a)^{n+1}$, and that for Problem 2.4 we have $p(z) \equiv 1$ and $q(z) = (z-a)(z-b)$.

4. Related problems

In class we have define what it means for two vector spaces to have the *same dimension*.

Problem 7. *Decide if $\mathbb{C}[z]$ and $\mathbb{C}(z)$, as vector spaces over $\mathbb{C}$, have the same dimension.*

Problem 8. *Do the same problem over $\mathbb{R}$ and over $\mathbb{Q}$.*

Problem 9. *Construct an explicit basis of $\mathbb{C}(z)$ over $\mathbb{C}$.*

Problem 10. *Do the same problem over $\mathbb{R}$ and over $\mathbb{Q}$.*

Good luck to you all!