Tsinghua Mathcamp Summer 2014

Research project: number theory

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This project is about several different way of expressing a real number. You are free to choose which problems in this project that you would like to work on, all I ask is that you work on things which you haven't seen before, and that you do not use any books or the internet.

Part 1: Upwards expansions

1. Basic upwards expansion

When we first met right after the orientation, Luchuan Yao suggested the following procedure: given a real number x_1 with $0 < x_1 < 1$, let $a_1 = [1/x_1] + 1$ and let $x_2 = a_1 x_1 - 1$. He then repeated this procedure, using x_2 in place of x_1, by solving the following problem:

Warm-up Problem 1.1. *Check that if $0 < x_1 < 1$ then the rules $a_n = [1/x_n] + 1$ and $x_{n+1} = a_n x_n - 1$ define numbers satisfying $0 < x_n \le x_{n-1} < \cdots \le x_2 \le x_1 < 1$ and $2 \le a_1 \le a_2 \le a_3 \le \ldots$*

We can use these numbers a_n and x_n to produce expressions

$$x_1 = \frac{1+x_2}{a_1} = \frac{1+\frac{1+x_3}{a_2}}{a_1} = \frac{1+\frac{1+\frac{1+x_4}{a_3}}{a_2}}{a_1} = \ldots$$

or equivalently

$$x_1 = \frac{1+x_2}{a_1} = \frac{1}{a_1} + \frac{1+x_3}{a_1 a_2} = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1+x_4}{a_1 a_2 a_3} = \ldots$$

Problem 1.2. *Show that for every positive real number x, there is a unique sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 \le a_2 \le a_3 \le \ldots$ and*

$$x = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \ldots$$

Problem 1.3. *Characterize rational numbers x in terms of the corresponding sequence $a_1, a_2, a_3, \dots$. That is, describe a property of the sequence $a_1, a_2, a_3, \dots$ which holds true if and only if x is rational. Do you recognize which number x corresponds to the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, ...?*

Now comes a more open-ended problem:

Problem 1.4. *For any $x > 0$, let $a_1, a_2, \dots$ be as in Problem 1.2. For each $n > 0$, write*

$$\frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \dots a_n}$$

as a fraction p_n/q_n where p_n and q_n are positive integers which are relatively prime to one another. "How well" does p_n/q_n approximate x? For instance, you can show that for any $\epsilon > 0$, if n is sufficiently large then

$$\left|x - \frac{p_n}{q_n}\right| < \frac{\epsilon}{q_n}.$$

But would the same conclusion be true if we replaced the right side by a function which approaches 0 more quickly, such as ${1/q_n}^2$? Try to produce the smallest such function you can, and in the other direction show that there are examples of x for which the function cannot be chosen to be too *small (in a sense that you will need to make precise).*

2. Upwards expansion with alternating signs

Problem 2.1. *For which real numbers x with $0 < x < 1$ does there exist an increasing sequence of positive integers $a_1 < a_2 < \dots$ such that*

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots?$$

Do any numbers have more than one such expansion?

Problem 2.2. *Solve the analogue of Problem 1.4 for the expansion in Problem 2.1. Is there a general result about whether one usually gets better rational approximations from the expansion in Problem 2.1 or the one in Problem 1.2? Is there any relationship between the two expansions?*

Problem 2.3. *Compute the first several terms of the expansion for* $x = \frac{3-\sqrt{5}}{2}$. *What patterns do you see? Try to make a conjecture about the exact values of the* a_i*'s, and then try to prove the conjecture. Is there a more general class of numbers which has a similar expansion as this special number?*

3. Nearest-integer upwards expansion

The procedure in Section 1 begins with the rule $a_n = [1/x_n] + 1$. RongShang Li suggested an alternate procedure, instead letting a_n be the integer closest to $1/x_n$ (breaking ties in any way you like).

Problem 3.1. *Give a precise definition of the* x_n*'s in this case, and build an expansion analogous to that in Problem 1.2. Solve the analogue of Problem 1.4 for this expansion. Does this usually give better rational approximations than the expansions in Problems 1.2 and 2.1?*

Part 2: Downwards expansions

4. General numbers

Given a real number x_0, define $a_0 := [x_0]$, and if $a_0 \neq x_0$ then put $x_1 := 1/(x_0 - a_0)$ and repeat this procedure with x_1 in place of x_0. As long as none of $x_0, x_1, \ldots, x_n$ are integers, this gives sequences defined by

$$a_k := [x_k] \quad \text{and} \quad x_{k+1} := \frac{1}{x_k - a_k} \quad \text{for } k = 0, 1, 2, \ldots, n-1.$$

Then

$$x_0 = a_0 + \frac{1}{x_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}} = \ldots$$

For each k, write the fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_k}}}$$

as the ratio p_k/q_k where p_k and q_k are coprime integers with $q_k > 0$. We call p_k/q_k the k-th convergent to the continued fraction expansion of x_0. Also we write $x_0 = [a_0; a_1, a_2, \ldots]$.

Problem 4.1. *Give a formula for p_k and q_k in terms of the a_i's. Can you give an efficient way to compute p_k in terms of a_k and the previous few p_i's?*

Problem 4.2. *Give a good upper bound on $|x_0 - p_k/q_k|$. Compare with the upper bounds that you or other students produced for rational approximations of x_0 in Part 1 of this project. Construct specific irrational numbers x_0 for which you can show that $|x_0 - p_k/q_k|$ is not much smaller than your general upper bound. Can you formulate a precise sense in which some specific collection of numbers x_0 are the irrational numbers which are approximated less-well by rational numbers than are any other irrational numbers?*

Problem 4.3. *Under what conditions do two numbers x_0 and y_0 have continued fraction expansions that end the same way, in the sense that $x_0 = [a_0; a_1, a_2, \ldots]$ and $y_0 = [b_0; b_1, b_2, \ldots]$ where there are some r and s so that $b_n = a_{r+n}$ for every $n > s$?*

5. Special numbers

Problem 5.1. *Compute the expansions $[a_0; a_1, a_2, \ldots]$ and the numbers x_k and p_k/q_k for each of $x_0 = \sqrt{2}$, $x_0 = \sqrt{3}$, $x_0 = \sqrt{5}$, $x_0 = \sqrt{6}$, and so on. Identify as many patterns as you can, make several general conjectures, and try to prove them. Be sure to examine $p_k^2 - d \cdot q_k^2$ where p_k/q_k are the convergents to the continued fraction expansion of $\sqrt{d}$.*

Problem 5.2. *Examine the continued fraction convergents of numbers of the form $x_0 := r/s$ where r is prime and $1 \leq s < r/2$. Do you see anything unusual? If so, formulate precise conjectures and then prove them.*