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A research project: Recurrence in Dynamics

Dmitry Kleinbock
Brandeis University

1. The set-up

The theory of dynamical systems studies objects that move around in space according to some rules, such as a gas particle in the air, or a billiard ball on a pool table, or a planet in the solar system – the list can be continued indefinitely. The lecture notes (Chapters 1 and 2) contain many examples, as well as describe connections of the subject with other areas of mathematics; we will discuss many examples and applications in the beginning of the course. The goal here is to study various properties of trajectories of the objects, and one particular phenomenon that is often present, and is quite important for applications, is **recurrence**, or, in plain terms, coming back to the same place. That is, if you hit a billiard ball starting from a point on one edge of the table, will it come back close to where it started? if yes, how soon? etc etc.

Formally, by a **dynamical system** we will mean a pair (X, T), where X is a metric space (a set with some notion of distance between points; those we will discuss in the beginning of the course) and $T : X \to X$ a rule determining how the points in X move in time. Starting from a point $x \in X$ one can consider trajectories

$$\{x, T(x), T\big(T(x)\big), \dots, T^n(x), \dots\}$$

and answer various questions about their behavior. A general overview of the subject can be found in Chapter 2 of the lecture notes.

Here is one important way a trajectory can behave: it may (or may not) keep coming back to its starting place. This gives rise to the concept of recurrence in dynamical systems, which goes back to the work of Poincare in the end of the XIX century. Specifically, a point x is said to be **recurrent** for T if for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $T^n(x) \in B(x, \varepsilon)$ (here and later $B(x, \varepsilon)$ stands for the open ball or radius ε centered at x). Equivalently, if the set of such n is infinite. Equivalently, if there is a sequence $n_k \to \infty$ such that $T^{n_k}(x) \to x$. (If these equivalences are not obvious to you, think of them as of an exercise).

The goal of this project will be to study some examples of dynamical systems and analyze their recurrence properties. Recurrence is the featured topic of Chapter 3 of the lecture notes; what you will be invited to do in this project is an in-depth study of some examples. The project description will be structured as a series of exercises, some easy, some more sophisticated. You do not have to do everything – choose the ones that you find attractive and challenging. How much you will do during the course depends on your interest.

2. A preliminary discussion

It is easy to come up with an example of a dynamical system which has no recurrent points. For instance, take $X = \mathbb{R}$ and $T : x \mapsto x + 1$, a right translation of the number axis by 1. Then all points will slide to the right and will never come back (that is, everything will go to infinity). This is not an interesting system to study, and during the course we will concentrate on spaces where going to infinity is not allowed. Such metric spaces will be called **compact**; we will discuss compactness in the beginning of the course. For now, just think of closed and bounded subsets of $\mathbb{R}$ or $\mathbb{R}^2$. For example, the unit interval $[0, 1] \subset \mathbb{R}$ (any other closed interval $[a, b]$ will do too).

Exercise 2.1. Let T be a continuous map from $[0,1]$ to itself. Show that it has a **fixed point**, that is, there exists $x \in [0,1]$ such that $T(x) = x$. Some knowledge of calculus is helpful here; you can visualize your proof by drawing a graph $\{y = T)x)\}$.

Clearly any fixed point is recurrent (because it does not ever move). Another example of recurrence is given by **periodic points**: x such that $T^n(x) = x$ for some n. Those are easy ways recurrence can show up. Sometimes it is the only way, as the next exercise shows:

Exercise 2.2. Construct a continuous map from $[0,1]$ to itself which has: (a) one fixed point and no other recurrent points; (b) n fixed points and no other recurrent points; (c) one fixed point, and the rest of the points are periodic. It may be helpful to draw sample graphs of some functions from $[0,1]$ to itself and try to visualize the behavior of trajectories using those graphs.

Note that continuity of the map T is important, as shown by the next exercise:

Exercise 2.3. Construct a discontinuous map T from $[0,1]$ to itself which has no recurrent points. Again, you can try to do it experimenting with graphs of various functions T. This can be achieved by creating just one point of discontinuity!

Thus in what follows, whenever we will talk about a dynamical system (X,T), we will take X to be compact and T continuous. Many interesting phenomena can still be observed! For example, one can discuss stability of fixed points: a fixed point x_0 is called **attracting** if $T^n(x) \to x_0$ whenever x is close enough to x_0, and **repelling** if there exists $\varepsilon > 0$ such that for any $x \in B(x_0, \varepsilon)$ different from x_0 there exists $n \in \mathbb{N}$ such that $T^n(x) \notin B(x_0, \varepsilon)$.

Exercise 2.4. Let T be a differentiable map from $[0,1]$ to itself and let x_0 be fixed by T. Prove that it is attracting if $|T'(x_0)| < 1$ and repelling if $|T'(x_0)| > 1$. Show that converse does not hold, that is, find examples of T and of attracting and repelling fixed points x_0 of T with $|T'(x_0)| = 1$. Also find examples of fixed points which are neither attracting nor repelling.

3. An interesting example: the tent map

This part is based on Exercise 2.14 of the lecture notes. Consider a continuous function T from $[0,1]$ to itself defined by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2; \\ 2 - 2x & \text{if } 1/2 \le x \le 1. \end{cases}$$

Exercise 3.1. Convince yourself that the name for this map is appropriate by drawing the graph of T.

Exercise 3.2. Find all the points $x \in [0,1]$ fixed by T.

Exercise 3.3. Now draw the graphs of T^2, T^3, ..., T^n. By looking at those graphs, determine the number of periodic points of T of period at most n.

Exercise 3.4. Prove that the set of periodic points of T is dense in $[0,1]$, that is, for any $y \in [0,1]$ there exists a sequence x_n of periodic points which converges to y.

Exercise 3.5. Explain what the map T does to x based on its binary expansion

$$x = 0.x_1x_2\dots \text{ where } x_i = 0 \text{ or } 1. \tag{3.1}$$

Note that the tent map is similar to the shift system described in section 2.4 of the lecture notes – yet is different from the shift.

Exercise 3.6. Use the previous exercise to prove that x is **eventually periodic** for T (that is, $T^n(x)$ is periodic for some $n \in \mathbb{N}$) if and only if x is rational.

Exercise 3.7. Construct x – in the form of its binary expansion (3.1) – which is recurrent for T but not periodic. See Exercise 4.10 of the lecture notes for a similar construction for the shift system – will it work here? This needs to be proved, or else the construction needs to be modified. Similarly, construct x which is not recurrent and not eventually periodic.

4. Another example: rotation of the circle

So far in all our examples of dynamical systems we were able to find fixed or periodic points. However fixed points do not always exist, even when X is compact and T is continuous. Yet it can be proved that recurrent points always exist: this is Birkhoff's Recurrence Theorem (Theorem 3.1 in the lecture notes) which we will prove during the course.

Here is a simple but very important example of a system without fixed points: just **rotate a circle**, viewed as a subset of $\mathbb{R}^2$, by some fixed angle (not a multiple of 2π). Clearly every point will move! Following the notation of Example 2.4 from the lecture notes, we are going to represent the circle S^1 by the interval $[0,1]$ with glued endpoints, or, equivalently, by the set of equivalence classes of points of $\mathbb{R}$, where two points a and b are equivalent if their difference is an integer (notation: $a \equiv b \mod 1$). The distance between two points $0 \leq x < y \leq 1$, viewed as points of S^1, is then given by the formula

$$d(x,y) \stackrel{\text{def}}{=} \min(y-x, x+1-y)\,.$$

Now, if α is a real number, we define the map $R_\alpha : S^1 \to S^1$ by $R_\alpha(x) = x + \alpha$, where $+$ is understood to be modulo 1 as well. Thus the full circle will correspond to rotation by 1, not by 2π. See Examples 2.4 and 2.10 from the lecture notes for more explanation and for another equivalent description.

Exercise 4.1. Show that all points of S^1 are periodic for R_α if α is rational, and no points are periodic otherwise. (This is jus a warm-up.)

Exercise 4.2. Show that $0 \in S^1$ is recurrent for R_α. (This has been already proved in class by students several times.)

Exercise 4.3. Derive from the previous exercise that every point of S^1 is recurrent for R_α.

Exercise 4.4. Even stronger, show that when α is irrational, every point of S^1 has a dense orbit. Equivalently, if I is a nonempty subinterval of S^1 and $x \in S^1$, then there exists $n \in \mathbb{N}$ such that $\mathbb{R}_\alpha^n(x) \in I$.

The latter property goes by the name of **minimality** and is discussed in great detail in Chapter 4 of the lecture notes, see in particular Proposition 4.2. Thus, irrational rotations are minimal. (Clearly every point in a minimal system is recurrent.)

Exercise 4.5. A higher-dimensional generalization: consider the k-dimensional torus $\mathbb{T}^k$, which is a direct product of k copies of S^1, or, equivalently, the set of equivalence classes of vectors of $\mathbb{R}^k$, where two vectors are equivalent if their difference is an integer vector. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and consider the map $R_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha}$ (here $\mathbf{x} = (x_1, \dots, x_k)$ and "$+$" means coordinate-wise addition modulo 1). Show that:

(a) $0 \in \mathbb{T}^k$ is recurrent for $R_{\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha} \in \mathbb{R}^k$; equivalently, if $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $\mathbb{R}_{\alpha_i}^n(0) \in B(0,\varepsilon)$ simultaneously for all $i = 1, \dots, k$;

(b) as before, (a) implies that all points of $\mathbb{T}^k$ are recurrent for $R_{\boldsymbol{\alpha}}$;

(c) $(\mathbb{T}^k, R_{\boldsymbol{\alpha}})$ is minimal if and only if the numbers $1, \alpha_1, \dots, \alpha_k$ are linearly independent over $\mathbb{Z}$; that is, if the equality $p + q_1\alpha_1 + \cdots + q_k\alpha_k = 0$ for integer $p, q_1, \dots, q_k \in \mathbb{Z}$ can only hold if $p, q_1, \dots, q_k = 0$ (this is a generalization of the concept of irrationality of a real number).

The proof should be a direct generalization of what you have done in the previous exercises.

Exercise 4.6. The previous exercises can be easily translated into the language of approximation of real numbers by rational ones. Indeed, we all know (or should know) that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$; that is, for every $\alpha \in \mathbb{R}$ and any $\varepsilon > 0$ there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $|\alpha - p/q| < \varepsilon$. However, Exercise 4.2 strengthens it into the following statement: for every $\alpha \in \mathbb{R}$ and any $\varepsilon > 0$ there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $|\alpha - p/q| < \varepsilon/q$. Write down analogous interpretations of Exercises 4.4 and 4.5.

Exercise 4.7. This exercise gives an unexpected application of Exercises 4.2 and 4.4.

(a) It is known that $2^4 = 16$ and $2^{10} = 1024$. Does there exist a power of 2 starting with 100? you can try to answer this question by playing with your calculator, but it will take an awfully long time! and what about starting with 1000? or with 1 followed by an arbitrary number of zeroes?

(b) It is not hard to check that any digit from 1 to 9 can appear as the first digit of a power of 2 (although you have to wait quite a bit until 7 and 9 appear. What about pairs of digits? here is the sequence

$$16, 32, 64, 12, 25, 51, 10, 20, 40, 81, 16, 32, 65, 13, 26, 52, 10, 20, 41, 83, \dots$$

of the first pairs of digits of powers of 2 starting with 16. Will any two-digit number eventually appear there? for example, is there a power of two starting with 99?

The answers to all the above questions are positive (in fact the same holds for k-tuples of first digits where $k \in \mathbb{N}$ is arbitrary), and your job is to connect this problem with circle rotations!

5. A quantitative approach to recurrence: return times

Let (X, T) be a dynamical system and let $x \in X$ be recurrent for T, which means that for any $\varepsilon > 0$ the trajectory of x eventually visits $B(x, \varepsilon)$. A natural question is – how long does it take x to come back to $B(x, \varepsilon)$? in other words, when $\varepsilon > 0$ and $x \in X$ one can define the quantity

$$\tau(x, \varepsilon) \stackrel{\text{def}}{=} \min\{n \in \mathbb{N} : T^n(x) \in B(x, \varepsilon)\},$$

which is called the **first return time** function. Clearly x is recurrent if and only if $\tau(x, \varepsilon) < \infty$ for any $\varepsilon > 0$. Also, x is a fixed point for T if and only if $\tau(x, \varepsilon) = 1$ for any positive ε.
For many dynamical systems it is possible to construct recurrent points with arbitrary large return times. The next exercise asks you to do it for the tent map:

Exercise 5.1. Let T be the tent map; prove that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$ there exists $x \in [0, 1]$ with $\tau(x, \varepsilon) > N$. That is, there exist points which return arbitrarily slowly. This is easy to do when T is just the shift of the binary digits of x, and only slightly trickier for the tent map which is a very close relative of the shift.

Now let us study return times for circle rotations. Turns out that the situation is very different there: there is a very explicit way to bound return times, uniform in all the starting points:

Exercise 5.2. Prove that for any $\alpha \in \mathbb{R}$, any $\varepsilon > 0$ and any $x \in S^1$ one has $\boxed{\tau(x, \varepsilon) \leq 1/\varepsilon}$ for $T = R_\alpha$ acting on S^1. In other words, all trajectories of circle rotations return fast enough. To do this, you need to simply revisit the work done in Exercises 4.2 and 4.3 and make the proofs quantitative.

Exercise 5.3. Use the previous exercise to give an upper bound, as a function of k, of the smallest n such that 2^n starts with 1 followed by k zeroes.

Exercise 5.4. Generalize Exercise 5.2 to the multidimensional set-up, that is, to rotations of the k-dimensional torus as in Exercise 4.5.

Note that the result of Exercise 5.2 is just an estimate, not equality: there are rotation angles which give rise to very fast return (for example, if α is rational, then we have $\tau(x, \varepsilon) \leq 1/\varepsilon$ The final part of the project asks you to show that it is not possible to significantly improve the conclusion of Exercise 5.2 for some rotation angles α:

Exercise 5.5. Take $\alpha = \sqrt{2}$. Then for any $0 < \varepsilon < 1$ and any $x \in S^1$ one has $\boxed{\tau(x, \varepsilon) \geq 1/3\varepsilon}$ for $T = R_\alpha$ acting on S^1. In other words, for this choice of α the return time is almost as big as it can possibly be (up to a constant)!

Here is the beginning of the argument. Take $x = 0$. Suppose the claim does not hold; that is, there exists $0 < \varepsilon < 1$ and $0 < n \leq 1/6\varepsilon$ such that $R^n_{\sqrt{2}}(0) \in B(0, \varepsilon)$, that is,

$$|n\sqrt{2} - m| < \varepsilon \leq \tfrac{1}{3n}$$

for some $m \in \mathbb{Z}$. Then multiply both sides of the above equation by $|n\sqrt{2} + m|$ and look for a contradiction...

As you can see, the argument boils down to studying the rate of approximation of $\sqrt{2}$ by rational numbers. With some work you can observe that the constant 1/3 can be slightly improved, and that similarly one can treat any α which is a root of a quadratic equation with integer coefficients. There also exist generalizations to higher degree algebraic numbers and to higher dimensions. All this belongs to the subject of **Diophantine approximation**, which has many interesting connections to dynamical systems.

6. Conclusion

There are many aspects of recurrence that one can study – for example, Chapter 3 gives an application of the existence of recurrent points to combinatorics Chapter 4 of the lecture notes discusses uniform recurrence, and in Chapter 5 recurrence is applied to studying small fractional parts of $n^2\alpha$ and other polynomials. Later chapters discuss multiple recurrence and its application to finding arithmetic progressions in large sets of integers.

The problems listed above give you some glimpses on the topic; what exactly you will choose for the project will be up to you, and will be subject to discussions with the course assistants. It is quite likely that as you work on the problems, new questions will pop up and you should feel free to think about them. If time allows, other examples of dynamical systems and more complicated problems can be added to the list to facilitate further research in this direction.

Good luck!