1. Functions

1.1. Graphs

Definition 1.1. *Let $S \subset \mathbf{R}^n$. An $\mathbf{R}^m$-valued function defined on S is a rule of assignment $f : S \to \mathbf{R}^m$, $X \mapsto f(X)$, which assigns to each point X in S a point $f(X)$ in $\mathbf{R}^m$; S is called the domain of f; we also write $D(f) = S$. The set of values $\{f(X)|X \in S\}$ of f is called the image of f and is denoted by $Im(f)$.*

• An $\mathbf{R}$-valued function defined on $S \subset \mathbf{R}^n$ is also called a scalar valued function of n variables.

• An $\mathbf{R}^m$-valued function $f : D(f) \to \mathbf{R}^m$ is nothing but a list $(f_1, .., f_m)$ of scalar valued functions $f_1, .., f_m$, defined on the same domain $D(f)$. Namely, for $X \in D(f)$, $f_i(X)$ is the ith component of the vector $f(X) \in \mathbf{R}^m$. Conversely, any given list of m scalar valued functions $f_1, .., f_m$ defined on the same domain $D(f_1) = \cdots = D(f_m)$, can be made into an $\mathbf{R}^m$-valued function f by setting $f(X) = (f_1(X), .., f_m(X))$. Therefore, any statement about $\mathbf{R}^m$-valued functions can be restated purely in terms of scalar valued functions, although not necessarily in a convenient form.

• A function is often specified by giving a formula for the value $f(X)$ in terms of the variables $X = (x_1, .., x_n)$. For example, the formula

$$f(x, y) = -x - y$$

defines a function on $\mathbf{R}^2$. Even though its domain is not explicitly named, it is clear that the formula makes sense for all $(x, y) \in \mathbf{R}^2$. Thus, we say in this case that $D(f) = \mathbf{R}^2$. For another example, the formula

$$f(x, y) = \frac{1}{x^2 + y^2}$$

defines a function on the punctured plane: $D(f) = \{X \in \mathbf{R}^2 | X \neq O\}$.

Definition 1.2. *The graph of an* $\mathbf{R}$*-valued function* $f : D(f) \to \mathbf{R}$ *is the set consisting of all points* $(x_1, .., x_n, z) \in \mathbf{R}^{n+1}$ *satisfying the equation* $z = f(X)$.

• In class, we sketch the graph of an affine function of two variables, i.e. a function of the form $f(x, y) = ax + by + c$.

• We also sketch the graph of the function $f(X) = x^2 + y^2$ by looking at its cross-sections (intersections) with various horizontal planes of the form $z = const.$ and with the planes $x = 0$ (the yz-plane) and $y = 0$ (the xz-plane.) In this case, the graph is a surface in $\mathbf{R}^3$ called a paraboloid.

• We also use the cross-section method to consider the function $f(X) = y^2 - x^2$. The graph in this case is a hyperboloid; it looks like a saddle. Likewise, the graph of the function $f(X) = y^2 - x$ looks like a gutter tilted at a slope.

• More generally, we consider the graph of the function $g(X) = ax^2 + bxy + cy^2 + dx + ey + f$, where $a, .., f \in \mathbf{R}$ are given constants. If we assume that $b^2 - 4ac \neq 0$, then this surface can always be transformed to one of the two "basic" surfaces, $z = x^2 + y^2$ or $z = y^2 - x^2$, above, by way of a combination of rigid rotations, reflections, coordinate scalings, and coordinate shifts (translation by a fixed vector.) In class, we sketch a proof of this involving the Spectral Theorem for 2×2 symmetric matrices, and a series of change of coordinates.

• Graphing is a useful, but somewhat crude, way to probe a function's properties. To do better, we need some analytical tools and notions, which we now begin to discuss.

1.2. Continuity

• Roughly speaking, we say that a function $f : D(f) \to \mathbf{R}$ is continuous at a point $X_0 \in D(f)$, if the value $f(X_0)$ *can be approximated* by the value $f(X)$ at *every* point X that is *nearby* X_0. In other words, we can make the "error" $|f(X) - f(X_0)|$ as small as we wish by "driving" X close enough to X_0. Since "error" can be a varying, we expect the "driver" to be a function of this variable error. It is customary to represent the varying error by the symbol ϵ and the driver $\delta(\epsilon)$. Here is the precise definition of continuity. Let $\mathbf{R}_>$ denote the set of all positive real numbers.

Definition 1.3. *We say that a function $f : D(f) \to \mathbf{R}$ is continuous at a point $X_0 \in D(f)$, if we can find a function $\delta : \mathbf{R}_> \to \mathbf{R}_>$ (we call a driver or a squeezing function,) such that for each $X \in D(f)$, the following statement is true:*

$$\|X - X_0\| < \delta(\epsilon) \Longrightarrow |f(X) - f(X_0)| < \epsilon.$$

If f is continuous at each point $X_0 \in D(f)$, we say that f is continuous.

• In a given situation, the choice of driver δ can depend on the point X_0 in question in a sensitive way. In class, we see this in the simple 1-variable example $f(x) = x^2$. The driver we use to show continuity at $x_0 = 0$ is $\delta(\epsilon) = \epsilon^{\frac{1}{2}}$. But this fails to be a driver at the point $x_0 = 1$. By reverse engineering, we see that one driver that works in this case is $\delta(\epsilon) = min(1, \epsilon/3)$.

Definition 1.4. *We say that an $\mathbf{R}^m$-valued function F is continuous at $X_0 \in D(F)$ if each of its components $f_1, .., f_m$ is continuous at X_0.*

• In class, we discuss the following theorem that allows us to deduce continuity of a function that is made up from some given continuous functions by algebraic operations: addition, multiplication, division and composition. The composition $g \circ K$ of two functions g, K is defined whenever, they are composable, i.e. $Im(K) \subset D(g)$. In this case, $(g \circ K)(X) = g(K(X))$ for $X \in D(K)$.

Theorem 1.5. *Let f, g be $\mathbf{R}$-valued functions defined on the same domain $D(f) = D(g)$, and let K be a $\mathbf{R}^n$-valued function with $Im(K) \subset D(g)$. If f, g are both continuous at X_0, then $f + g$ and fg are also continuous at X_0. If f is continuous at X_0 and $f(X_0) \neq 0$, then $1/f$ is also continuous at X_0. If K is continuous at $Y_0 \in D(K)$ and if g is continuous at $K(Y_0)$, then $g \circ K$ is also continuous at Y_0.*

• In class, we prove the continuity of $f + g$ at X_0 by writing down a driver δ explicitly in terms of two given drivers δ_1, δ_2, for f and g. Namely $\delta(\epsilon) = min(\delta_1(\epsilon/2), \delta_2(\epsilon/2))$. There are similar formulas in the cases of fg, $1/f$ and $g \circ K$, but are much more complicated and won't be needed in the rest of the course.

• Examples. Any constant function f is continuous. In fact, it is evident that constant function $\delta(\epsilon) = 1$ will work as a driver in this case, regardless of the point $X_0 \in D(f)$.

Theorem 1.6. *Any linear function on $\mathbf{R}^n$ is continuous.*

Proof: A linear function is of the form $f(X) = A \cdot X$ where $A \in \mathbf{R}^n$ is some fixed (i.e. independent of X) vector. If $A = O$, then f is constant function, which we know is continuous. Consider the case $A \neq O$. Given $X_0 \in \mathbf{R}^n$, we want to find a driver $\delta(\epsilon)$ that ensures $|A \cdot X - A \cdot X_0| < \epsilon$. By the Schwarz inequality, this holds if we can make $|X - X_0| < \epsilon/\|A\|$. But we can make the last inequality hold if we choose the driver $\delta(\epsilon) = \epsilon/\|A\|$. Thus we have found a desired driver. □

• Example. A monomial function (of n variables) $f : \mathbf{R}^n \to \mathbf{R}$ is one of the form $f(x_1, .., x_n) = ax_{i_1} \cdots x_{i_k}$, where a is a given constant and $i_1, .., i_k$ are given indices, not necessarily distinct. For example, $f(x, y, z) = 2x^2 y$ is a monomial function of 3 variables. A constant function is also regarded as a monomial function. A polynomial function is a sum of finite number of monomial functions. Thus any linear function is an example of a polynomial function.

Theorem 1.7. *Any polynomial functions is continuous.*

Proof: Note that a monomial function is just a product of linear functions. Since we know that linear functions are continuous and that a product of continuous functions is continuous, any monomial function is continuous. We also know that any (finite) sum of continuous function is continuous. Since a polynomial function is a sum of monomial functions, it too is continuous. □

• In class, we recall from 1-variable calculus that the function $g(t) = t^{\frac{1}{2}}$ defined on $\mathbf{R}_+$ is continuous. We know that the polynomial function $f(X) = \|X\|^2 = x_1^2 + \cdots + x_n^2$ is continuous on $\mathbf{R}^n$. Thus the composition given by $(g \circ f)(X) = \|X\|$ is also continuous. This is called the length function.

• Let $h : \mathbf{R} \to \mathbf{R}$ be any 1-variable continuous function. Then the composition given by

$$[h \circ (g \circ f)](X) = h(\|X\|)$$

is also continuous. This shows that any continuous function of the length $\|X\|$ gives a continuous function of n variables.

2.1. Limit Points and Limits

• Roughly speaking, a limit point of a set $S \subset \mathbf{R}^n$ is a point which can be approximated by members of S. More precisely,

Definition 1.8. *We say that $X_0 \in \mathbf{R}^n$ is a limit point of S if for every $\epsilon > 0$, we can find a point $X \in S$ that is within distance ϵ from, but not equal to, X_0, i.e. $0 < \|X - X_0\| < \epsilon$.*

• An equivalent way to state this is by using the notion of a ball. A *ball* of radius $r > 0$ centered at X_0 in $\mathbf{R}^n$ is the set

$$B(X_0; r) = \{X \in \mathbf{R}^n | \|X - X_0\| < r\}.$$

Thus, a point $X_0 \in \mathbf{R}^n$ is a limit point of S iff every ball $B(X_0; \epsilon)$ centered X_0 contains a member of S other than X_0, i.e. iff for every $\epsilon > 0$, the set $S \cap B(X_0; \epsilon)$ contains a point $X \neq X_0$.

• For example, consider the unit ball $S = B(O; 1)$ in $\mathbf{R}^2$; this is the set of all points whose distance from the origin O is < 1. Intuitively, it is evident that a point that can be approximated by points in S is either in S or on the boundary circle. In other words, it should be true that $X_0 \in \mathbf{R}^n$ is a limit point of S iff $\|X_0\| \leq 1$. We prove this by proving the following two statements:

(1) If $\|X_0\| > 1$, then X_0 is not a limit point of S.

(2) If $\|X_0\| \leq 1$, then X_0 is a limit point of S.

• To prove (1), assume $\|X_0\| > 1$. Then the ball $B(X_0; \|X_0\| - 1)$ contains no points of S. This implies that X_0 is not a limit point of S. To prove (2), assume $\|X_0\| \leq 1$; if $X_0 = O$, then $S \cap B(X_0; \epsilon)$ contains the point $X = \frac{1}{2} min(1, \epsilon) E_1 \neq O$; if $X_0 \neq 0$, then a picture tells us that we should try to find a point in $X = kX_0 \in S \cap B(X_0; \epsilon)$ along the line segment connecting O, X_0. To make $X \neq X_0$, we impose $0 \leq k < 1$. This implies $X \in S$ as well, since $\|X_0\| \leq 1$. To make $X \in B(X_0; \epsilon)$, we impose $1 - k < \epsilon$. We can find a number k that fulfills the two requirements we impose; for example, $k = max(1 - \frac{1}{2}\epsilon, \frac{1}{2})$.

• Already in this simple example, while guessing by intuition the limit points of S is easy, writing a proof that the guess is correct takes some getting use to the analytical language, though none of the steps are difficult. This is typical of working with analytical notions such as limit points.

• The following definition evidently resembles the definition of continuity. But there are important differences between the two which we explain later along with their relationship.

Definition 1.9. *Let f be an $\mathbf{R}$-valued function and X_0 a limit point of its domain $D(f)$. We say that a number t is a limit of $f(X)$ as X approaches X_0, and we write "$f(X) \to t$*

as $X \to X_0$" or simply "$lim_{X \to X_0} f(X) = t$", if we can find a function $\delta : \mathbf{R}_> \to \mathbf{R}_>$ (we call a driver,) such that for each $X \in D(f)$, the following statement is true:

$$0 < \|X - X_0\| < \delta(\epsilon) \Longrightarrow |f(X) - t| < \epsilon.$$

• If $F = (f_1, .., f_m)$ is a $\mathbf{R}^m$-valued function and $X_0 \in D(F)$, we say that $lim_{X \to X_0} F(X) = T$ iff $lim_{X \to X_0} f_i(X) = t_i$; here $T = (t_1, .., t_m)$.

• In class, we prove that if a limit of a function exists at a point X_0, then the limit is unique, so that we can speak of *the* limit of a function at a point.

Theorem 1.10. *(Uniqueness of limit) If $f(X) \to t$ as $X \to X_0$ and if $f(X) \to s$ as $X \to X_0$, then $s = t$.*

• There are important differences between the continuity and the limit definitions. For the continuity definition, the point in question X_0 is required to be in the domain $D(f)$ so that $f(X_0)$ makes sense, and X_0 need not be a limit point of $D(f)$. In the limit definition, whether or not X_0 is in $D(f)$ is irrelevant, but X_0 is required to be a limit point of $D(f)$.

• Here is an example to illustrate this. Consider the functions

$$f_1 : \mathbf{R} \to \mathbf{R}, \quad f_1(x) = x$$
$$f_2 : \mathbf{R} \backslash \{0\} \to \mathbf{R}, \quad f_2(x) = x.$$

Note that 0 is a limit point of both $D(f_1), D(f_2)$. f_1 is continuous at 0 and $f(x) \to 0$ as $x \to 0$. But f_2 is not defined at the point 0, and so we cannot ask if f_2 is continuous at f_2. Nevertheless, we have $f_2(x) \to 0$ as $x \to 0$.

Theorem 1.11. *(Continuity-Limit) Let f is a $\mathbf{R}$-valued function and X_0 be both a limit point of and a member of $D(f)$. Then f is continuous at X_0 iff $lim_{X \to X_0} f(X) = f(X_0)$.*

This is an immediate consequence of the continuity and the limit definitions. This theorem gives a quick way to find the limit at a point if one knows that f is continuous at that point.

• Let f be a polynomial function of n variables and let $X_0 \in \mathbf{R}^n$ be any given point. Does the limit $lim_{X \to X_0} f(X)$ exist? If so, what is its value? We know that polynomial functions are continuous. By the Continuity-Limit theorem, the limit we seek exists and is equal to $f(X_0)$.

• Not surprisingly, the same rules that govern sums, products, reciprocals and compositions of continuous functions also hold for limits. For example, if $f(X) \to t$ and $g(X) \to s$ as $X \to X_0$, then $(f+g)(X) \to t+s$ as $X \to X_0$.

• Example. Consider $f(X) = \frac{xy}{(x^2+y^2)^{\frac{1}{2}}}$ defined for $X \neq O$ in $\mathbf{R}^2$. Where is f continuous? O is a limit point of $D(f)$. Does $f(X)$ have a limit as $X \to O$? First, xy is monomial function, hence is continuous. We know that the length function is continuous everywhere and vanishes only at the origin, so $1/\|X\| = 1/(x^2+y^2)^{\frac{1}{2}}$ is continuous away from the origin. It follows that f is a product of two continuous functions, so it too is continuous on the punctured plane $D(f)$; O is a limit point of $D(f)$. We claim that $f(X) \to 0$ as $X \to O$. To see this, we use the following theorem.

1.3. Squeeze Theorem

Theorem 1.12. *(Squeeze theorem) Let f, g be $\mathbf{R}$-valued functions defined on the same domain $D(f) = D(g)$, and let X_0 be a limit point of this set. Suppose that $|f(X)| \leq |g(X)|$ for each $X \in D(f)$. If $g(X) \to 0$ as $X \to X_0$, then $f(X) \to 0$ as $X \to X_0$.*

Proof: Since g has the limit 0 at X_0, we can find a driver δ such that for $X \in D(g)$,

$$0 < \|X - X_0\| < \delta(\epsilon) \Longrightarrow |g(X)| < \epsilon.$$

The same driver works for f as well, since $|g(X)| < \epsilon \Longrightarrow |f(X)| < \epsilon$ by our supposition. This shows that f has the limit 0 at X_0. □

Returning to the preceding example, our key observation is that $|y/(x^2+y^2)^{\frac{1}{2}}| \leq 1$. It follows that $|f(X)| = |x| \; |y/(x^2+y^2)^{\frac{1}{2}}| \leq |x|$ for each $X \neq O$. Thus we can apply the preceding theorem to the functions f and $g(x, y) = x$. Since $g(X) \to 0$ as $X \to O$, it follows that $f(X) \to 0$ as $X \to O$.

Note that we can now extend f to a new continuous function $\tilde{f} : \mathbf{R}^2 \to \mathbf{R}$, which has a larger domain and which has the same values as f on $D(f)$. Namely, $\tilde{f}(X) = f(X)$ for $X \in D(f)$, and $\tilde{f}(O) = 0$. Since

$$lim_{X \to O} \tilde{f}(X) = lim_{X \to O} f(X) = 0 = \tilde{f}(O),$$

$\tilde{f}$ is continuous at O, by the Continuity-Limit theorem.

1.4. Two-path Test

Definition 1.13. *A (continuous) path or a parametrized curve in $\mathbf{R}^n$ is a continuous $\mathbf{R}^n$-valued function $c : I \to \mathbf{R}^n$; here I is an interval in $\mathbf{R}$. We say that the curve passes through X_0 exactly at time $t_0 \in I$ if $c(t_0) = X_0$ and if $c(t) \neq X_0$ for $t \neq t_0$.*

• Example. $c(t) = (cos\ t, sin\ t)$ defines a path that winds around the unit circle in $\mathbf{R}^2$. If we restrict t to small interval, say $-\pi/4 < t < \pi/4$, then c passes through $(1, 0)$ exactly at time $t = 0$.

• Example. In Math 22a, we saw that if A, B are fixed vectors in $\mathbf{R}^n$, $c(t) = A + tB$ defines a parametrized line parallel to the vector B and passes through the point A exactly at time $t = 0$.

Theorem 1.14. *(Path Independence theorem) Let f be a $\mathbf{R}$-valued function and A a limit point of $D(f)$. Let c be any path that passes through A exactly at time $t = 0$; assume that $c(t) \subset D(f)$ for $t \neq 0$. If $lim_{X \to A} f(X) = s$, then $lim_{t \to 0} f(c(t)) = s$.*

Proof: We include a proof here for those who wish to see the analytical details.

Since f has s as a limit at A, we can find a driver δ' such that for $X \in D(f)$,

$$0 < ||X - A|| < \delta'(\epsilon) \Longrightarrow |f(X) - s| < \epsilon.$$

Write $c(t) = (x_1(t), .., x_n(t))$. By continuity of c, we can find drivers δ_i, $i = 1, .., n$, such that

$$0 < |t| < \delta_i(\epsilon) \Longrightarrow |x_i(t) - a_i| < \epsilon.$$

We try to reverse engineer a driver δ that we need, as follows:

$$\begin{aligned}
&|f(c(t)) - s| < \epsilon \\
\Longleftarrow\ & 0 < ||c(t) - A|| < \delta'(\epsilon) \\
\Longleftarrow\ & c(t) \neq A \ \ \& \ \ |x_i(t) - a_i| < \delta'(\epsilon)/\sqrt{n} \ \ \forall i \\
\Longleftarrow\ & 0 < |t| < \delta_i(\delta'(\epsilon)/\sqrt{n}) \ \forall i.
\end{aligned}$$

So put

$$\delta(\epsilon) = min(\delta_i(\delta'(\epsilon)/\sqrt{n}), \ i = 1, .., n).$$

This completes the proof. □

Applying the contra-positive of this, we get

Corollary 1.15. *(Two-path Test) Let f be a $\mathbf{R}$-valued function and A a limit point of $D(f)$. If we can find two paths c and d that pass through A exactly at time $t = 0$ and have $c(t), d(t) \in D(f)$ for $t \neq 0$, and if the limits $lim_{t\to 0} f(c(t))$ and $lim_{t\to 0} f(d(t))$ exist but are not equal, then $f(X)$ has no limit at A.*

• Example. Consider $f(X) = \frac{xy}{x^2+y^2}$, defined on the punctured plane. O is a limit point of $D(f)$. For any unit vector A, the path tA in $\mathbf{R}^2$ passes through O exactly at time $t = 0$. We have $lim_{t\to 0} f(tE_1) = 0 = lim_{t\to 0} f(tE_2)$. But we can't make any conclusion about the existence of $lim_{X\to O} f(X)$. Now $lim_{t\to 0} f(tE_1 + tE_2) = lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}$. So we have found two paths that yield two different limits, as the Two-path test calls for, and we can conclude that $f(X)$ has no limit at O.

• Although the preceding example closely resembles the function $\frac{xy}{(x^2+y^2)^{\frac{1}{2}}}$ we saw earlier, the two behave very differently at O: the former cannot be extended to a continuous function on $\mathbf{R}^2$, while the latter can, because one has a limit and the other doesn't at O.

• Example. Testing the limits of a function along a limited type of paths can be misleading. In this example, $f(c(t))$ has the same limit for *every line* c passing through O in $\mathbf{R}^2$, from which one may be tempted to think that $f(X)$ has a limit at O. But it turns out to have a different limit for a certain curve, which shows that f is actually discontinuous at O. Consider

$$f(X) = \frac{2xy^2}{x^2 + y^4}.$$

Then for any nonzero vector A, $f(tA) \to 0$ as $t \to 0$. But for the parabolic path $c(t) = (t^2, t)$, the function $f(c(t)) = 2t^4/2t^4 = 1$ has the limit 1 at $t = 0$. Thus, by the Two-path test, $f(X)$ has no limit at O.

1.5. Interior Points

Definition 1.16. *Let S be a set in $\mathbf{R}^n$. We call $X_0 \in S$ an interior point of S if S contains a ball $B(X_0, \epsilon)$, centered at X_0. The set of interior points of S is called the interior of S, denoted by $Int(S)$.*

• Since a ball is an infinite set, if a set S has an interior point, then S is necessarily an infinite set. Thus a finite set can never have an interior point.

• Consider the set $S = B(O; 1)$. Since $B(O; 1) \subset B(O; 1)$, O is an interior point of S. If $X_0 \in S$ is $\neq O$, then $B(X_0; \epsilon) \subset B(O; 1)$ for $\epsilon = min(1 - \|X_0\|, \|X_0\|)$, so X_0 is an interior point of S. Thus every point of S is in its interior. Thus $S = Int(S)$ in this case.