

NAME:

Write your answers in the space provided. Do not hand in loose sheets. You are allowed a letter size two-sided aid sheet. You may use any facts stated in the text book or in class. You may also use any facts about the real numbers and any results from one-variable calculus.

1. [35] Consider the iterated integral

$$J = \int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} f(x, y) dx dy$$

where  $f(x, y) = 4x^2 + 3x + 4y^2$ .

(a) [15] Interpret  $J$  as the integral of  $f$  over a region  $D$  in the  $xy$ -plane by drawing a picture of  $D$ . You must clearly identify the boundary of  $D$ .

(b) [20] Use (a) and an appropriate version of Fubini's theorem to compute  $J$ .

(a)  $D$  is the region sandwiched between the graphs of the two functions of  $y$ , namely  $y$  and  $\sqrt{1-y^2}$ , defined on the interval  $[0, \frac{1}{\sqrt{2}}]$ . In other words,

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{1}{\sqrt{2}}, y \leq x \leq \sqrt{1-y^2}\}.$$

(b)  $D$  can be viewed as the fan-like region between two radial lines  $\theta = 0$  and  $\theta = \pi/4$ , and two circles  $r = 0$ ,  $r = 1$ . Thus

$$J = \int_{\theta=0}^{\pi/4} \int_0^1 (4r^2 + 3r \cos \theta) r dr d\theta.$$

Performing the two 1-variable integrals, we get  $J = \frac{\pi}{4} + \frac{1}{\sqrt{2}}$ .

2. [30] Consider the set  $S$  defined by the equation  $x^2 + y^2 - z^2 - 1 = 0$  in  $\mathbb{R}^3$ .
- (a) [10] Without using a parameterization, prove that  $S$  is smooth of dimension 2, i.e. it is a surface.
- (b) [5] Draw a picture of  $S$ .
- (c) [15] Write down an explicit parameterization  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  for this surface, compute the Jacobian matrix  $F'(X)$ , and verify that it has rank 2. (Hint: Use trigonometric functions.)

(a)  $S$  is the zero level set of  $g(X) = x^2 + y^2 - z^2 - 1$ . Since  $g'(X) = (2x, 2y, -2z) \neq O$  on  $S$ , applying our sufficient condition for surface, we find that  $S$  is a surface.

(b) Note that for each fixed value of  $z = a$ ,  $x^2 + y^2 = 1 + z^2$  is a circle in the plane  $z = a$  of square radius  $1 + a^2$ , and centered at  $(0, 0, a)$ . So  $S$  can be viewed as a family of circles, one for each  $a \in \mathbb{R}$ . If we look at the cross-section of  $S$  with the  $xz$ -plane, we find the hyperbola  $x^2 - z^2 = 1$ .  $S$  can be obtained as a surface of revolution by rotating the hyperbola round the  $z$ -axis.

(c) Since  $\cos^2 \theta + \sin^2 \theta = 1$  for  $\theta \in \mathbb{R}$ , we have

$$(\sqrt{1 + z^2} \cos \theta)^2 + (\sqrt{1 + z^2} \sin \theta)^2 = 1 + z^2.$$

Thus the function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $F(\theta, z) = (\sqrt{1 + z^2} \cos \theta, \sqrt{1 + z^2} \sin \theta, z)$ , maps  $\mathbb{R}^2$  into  $S$ . In fact,  $Im(F) = S$ .

$F'(X)$  is a  $3 \times 2$  matrix with columns  $\frac{\partial F}{\partial \theta}(X)$ ,  $\frac{\partial F}{\partial z}(X)$ . We have

$$\frac{\partial F}{\partial \theta}(X) = (-\sqrt{1 + z^2} \sin \theta, \sqrt{1 + z^2} \cos \theta, 0)$$

$$\frac{\partial F}{\partial z}(X) = \left( \frac{z \cos \theta}{\sqrt{1 + z^2}}, \frac{z \sin \theta}{\sqrt{1 + z^2}}, 1 \right)$$

Consider a general linear relation of the two vectors  $a \frac{\partial F}{\partial \theta}(X) + b \frac{\partial F}{\partial z}(X) = O$ . The third component implies that  $b = 0$ . Since  $\|\frac{\partial F}{\partial \theta}(X)\|^2 = 1 + z^2 > 0$ , it follows that  $a = 0$ . Thus the two columns of  $F'(X)$  are independent, hence it has rank 2.

3. [35] You are asked to design an oval-shaped swimming pool. The pool's bottom which is level has a fixed area of  $A$  ft<sup>2</sup>. There are 4 segments to the pool's wall: the north and south walls are straight, while east and west walls are semi-circular. The costs of tiling the walls are:  $s$  dollars/ft for the straight ones,  $c$  dollars/ft for the semi-circular ones. Assuming  $c > s > 0$ , use the Lagrange multiplier method to find the lengths of those walls which gives a critical point for the cost function. Your answers should be in terms of the given constants  $A, c, s$ . (As a check, when  $A = 1600$ ,  $c = 2$ ,  $s = 1$ , then lengths of the straight and semi-circular edges are  $40\sqrt{\frac{\pi}{3}}$  ft.)

Name the lengths of the east wall  $x$  and of south wall  $y$  (draw a picture!) Then the cost function is

$$f(x, y) = 2cx + 2sy.$$

The area constraint is

$$g(x, y) := \pi\left(\frac{x}{\pi}\right)^2 + 2\frac{x}{\pi}y - A = 0.$$

Note that the radius of the semi-circular wall is  $x/\pi$ . We want to find critical points of  $f$  under this constraint by the Lagrange multiplier method. That means solving the equations

$$\nabla f + \lambda \nabla g = 0, \quad g = 0.$$

There is a unique solution given by

$$x = \frac{\pi s \sqrt{A}}{\sqrt{2\pi s c - \pi s^2}}, \quad y = \frac{\pi(c - s)\sqrt{A}}{\sqrt{2\pi s c - \pi s^2}}.$$

4. (Bonus) Prove that every increasing function  $f : [0, 1] \rightarrow \mathbb{R}$ , defined on the interval  $[0, 1]$ , is integrable. Here “increasing” means that  $1 \geq t_1 > t_2 \geq 0 \implies f(t_1) \geq f(t_2)$ .

Let  $P_N$  be the partition consisting of the intervals  $[0, \frac{1}{N}]$ ,  $[\frac{1}{N}, \frac{2}{N}]$ ,  $\dots$ ,  $[\frac{N-1}{N}, 1]$ . Since  $f$  is increasing, it follows that

$$\sup f([\frac{i}{N}, \frac{i+1}{N}]) = f(\frac{i+1}{N}), \quad \inf f([\frac{i}{N}, \frac{i+1}{N}]) = f(\frac{i}{N}).$$

It follows that

$$U(P_N) - L(P_N) = \frac{1}{N}(f(1) - f(0)).$$

But this goes to 0 as  $N \rightarrow \infty$ . By the Integrability Criterion, it follows that  $f$  is integrable.