Math 23b **Homework 1** **Spring, 2009**

Due Wednesday, January 28

Be sure to write clearly, using complete sentences. Do not use symbols such as $\forall$, $\exists$, $\rightarrow$, $\Rightarrow$, $\Rightarrow\!\Leftarrow$, $\therefore$, , etc., and do not use abbreviations like s.t., w/, w/o, b/c, c/o, etc. You may use the membership notation such as $a \in \mathbb{R}$.

1. Let $A = \{\, j^2 - j \mid j \in \mathbb{Z} \,\}$ and let $B = \{\, k^2 + k \mid k \text{ is a nonnegative integer} \,\}$. Prove that $A = B$.

2. Let $A = \{\, x^2 - y^2 \mid x, y \in \mathbb{Z} \,\}$ and let B be the set of integers that are either odd or divisible by 4. Prove that $A = B$. (Hint: Note that $x^2 - y^2 = (x - y)(x + y)$.)

3. Determine the set of solutions to the inequality

$$\left| \frac{x}{x+1} \right| \leq 1.$$

You must prove that your answer is correct. Note that this problem involves showing that two sets are equal: one set is the set of solutions of this inequality, and the other is the set that you claim it is. (Hint: One approach is to consider several cases. Another approach involves squaring both sides.)

4. Let x, y, u, and v be real numbers. Prove that $(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)$. (Hint: You can prove $A \leq B$ by proving $B - A \geq 0$.)

5. The arithmetic-geometric mean inequality for n variables (AGM_n) is the inequality

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers. In class we saw that AGM_n is true for $n = 2$, 3, and 4. Prove that if AGM_n is true then AGM_{2n} is true.