NAME:

Be sure to write clearly, using complete sentences, in the space provided. Do not hand in loose sheet. Do not use abbreviations like s.t., w/, w/o, b/c, c/o, etc. You are allowed a letter size two-sided aid sheet. You may use any facts in the book or in the lecture notes. You have 50 minutes.

1. [30] Determine the set of solutions to the inequality

$$\left|\frac{x-3}{x+5}\right| \leq 1.$$

You must prove that your answer is correct. Note that this problem involves showing that two sets are equal: one set is the set of solutions of this inequality, and the other is the set that you claim it is.

For $x \in \mathbb{R}$, the following are equivalent:

$$\left|\frac{x-3}{x+5}\right| \leq 1$$

$$(x-3)^2 \leq (x+5)^2$$

$$-6x+9 \leq 10x+25$$

$$-1 \leq x.$$

It follows that

$$\{x \in \mathbb{R} | \left|\frac{x-3}{x+5}\right| \leq 1\} = \{x \in \mathbb{R} | -1 \leq x\}$$

because the membership tests for the two sets have been shown to be equivalent.

2. [30] Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $\mathbb{Z}$ be the set of integers. For each of the following verbal statements, give its symbolic form, negate this symbolic statement, and then write the negated statement in verbal form without using words of negation or symbols such as $\neq, \neg$.

(a) For any real number y and integer x, if $x + 1 > y > x$ then $f(x) > f(y)$.

$$(\forall y \in \mathbb{R})(\forall x \in \mathbb{Z})(x + 1 > y > x \;\Rightarrow\; f(x) > f(y)).$$

Since $\neg(P \Rightarrow Q) = \neg(\neg P \vee Q) = P \wedge \neg Q$, the negation is

$$(\exists y \in \mathbb{R})(\exists x \in \mathbb{Z})(x + 1 > y > x \;\wedge\; f(x) \leq f(y)).$$

This says that there is a real number y and an integer x, such that $x + 1 > y > x$ and $f(x) \leq f(y)$.

(b) For some real number x such that $x > 0$, we have $f(x) > 0$.

$$(\exists x \in \mathbb{R})(x > 0 \;\wedge f(x) > 0).$$

By one of de Morgan's laws, the negation is

$$(\forall x \in \mathbb{R})(x \leq 0 \vee f(x) \leq 0).$$

This says that for any real number x, we have $x \leq 0$ or $f(x) \leq 0$.

(c) We can expressed every integer as $f(x)$, for some real number x.

$$(\forall y \in \mathbb{Z})(\exists x \in \mathbb{R})(f(x) = y).$$

The negation is

$$(\exists y \in \mathbb{Z})(\forall x \in \mathbb{R})(f(x) > y \vee f(x) < y).$$

This says that there is an integer y such that for any real number x, we have $f(x) > y$ or $f(x) < y$.

3. [30] Complete the last two columns of the following truth table:

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(a) What can you say about the last two columns?
They are the same.

(b) Let A, B, C be sets. Draw two separate Venn diagrams, one for each of the two sets $(A \cup B) - C$ and $[A - (B \cup C)] \cup [B - (A \cup C)]$.

(c) Circle the FIRST false statement in the following wrong proof of the following assertion:

$$(*) \quad (A \cup B) - C \;\subset\; [A - (B \cup C)] \cup [B - (A \cup C)].$$

In no more than two lines explain why the statement you circle is false.

Proof: Let x be a member of the left side of (*). Then $x \notin C$. In addition, $x \in A$ or $x \in B$ is true. We will try to show that in either case, x is a member of the right side of (*).
Since $x \notin C \subset B \cup C$, it follows that $x \notin B \cup C$. Likewise, $x \notin A \cup C$.
In the first case, we have $x \in A$. Since $x \notin B \cup C$, it follows that $x \in A - (B \cup C)$. So, x is a member of the right side of (*) in this case.
In the second case, we have $x \in B$. Since $x \notin A \cup C$, it follows that $x \in B - (A \cup C)$. So, x is a member of the right side of (*) in this case as well.

The statement "Since $x \notin C \subset B \cup C$, it follows that $x \notin B \cup C$" is the first false statement in the argument. Note that $x \notin C$ and $C \subset B \cup C$ do not imply $x \notin B \cup C$, in general.

4. [10] Let $I = \{x \in \mathbb{R} |\ |x| < 1\}$. Define the function $f : I \to \mathbb{R}$, $f(x) = \frac{x}{1-x^2}$. Use the Bijectivity Test to prove that f is bijective.

Let's discover the inverse of f first. For a given $x \in I$, put $y = f(x)$. Then

$$yx^2 + x - y = 0.$$

Note that $x = 0$ if and only if $y = 0$. For $x \neq 0$, solving this quadratic yields two possible choices:

$$x = -\frac{1}{2y} \pm \sqrt{1 + \frac{1}{4y^2}}.$$

Observe that y must have the same sign as x, so the sign $\pm$ must be chosen according to this. The result is $\frac{-1+\sqrt{1+4y^2}}{2y} = \frac{2y}{1+\sqrt{1+4y^2}} = x$. Note that this holds for $x = y = 0$ as well. So, our candidate for the inverse function $g : \mathbb{R} \to I$, should be defined by

$$g(y) = \frac{2y}{1 + \sqrt{1 + 4y^2}}.$$

Note that $|g(y)| < 1$ for all $y \in \mathbb{R}$. Now we verify that for $x \in I$, $y \in \mathbb{R}$,

$$g(f(x)) = x, \quad f(g(y)) = y.$$

The above calculation shows that

$$y = f(x) \Leftrightarrow x = \frac{2y}{1 + \sqrt{1 + 4y^2}} = g(y).$$

This shows that $g(f(x)) = x$ for all $x \in I$. The second equation can be verified similarly. You can also verify the two equations by direct substitutions.