Math 23b **Quiz 2** **Spring, 2009**

NAME:

Be sure to write clearly, using complete sentences, in the space provided. Do not hand in loose sheet. You are allowed a letter size two-sided aid sheet. You may use any facts in the book or proved in class or homework assignments. You have 50 minutes. Since each problem has restriction on how much you can write, you should consider carefully what is essential to include, before writing your final answers.

1. [35] Let $\langle a \rangle$ be a sequence of numbers such that $a_0 = 1$ and $a_n = 5a_{n-1} + 1$ for $n > 0$. In no more than 10 lines, prove that $a_n = \frac{1}{4}(5^{n+1} - 1)$ for all $n \geq 0$.

Do induction on $n \geq 0$. We have $a_0 = 1 = \frac{1}{4}(5^{0+1} - 1)$, so the assertion holds for $n = 0$. Suppose the assertion holds for $n = k$. That is, $a_k = \frac{1}{4}(5^{k+1} - 1)$. So,

$$a_{k+1} = 5a_k + 1 = \frac{1}{4}(5^{k+2} - 5) + 1 = \frac{1}{4}(5^{k+2} - 1).$$

So the assertion holds for $n = k + 1$ as well.

2. [35] Let $a_1 = 1$, $a_2 = 2$, and $a_{n+2} = \frac{1}{2}(a_{n+1} + 3/a_n)$ for $n \geq 3$. In no more 10 lines, prove that $1 \leq a_n \leq 3$ for $n \in \mathbb{N}$.

We will use strong induction. We have $1 \leq a_1, a_2 \leq 3$. Take $n \in \mathbb{N}$, and assume $1 \leq a_n, a_{n+1} \leq 3$. It follows that

$$1 \leq a_{n+1} \leq 3, \quad 1 \leq 3/a_n \leq 3.$$

Thus

$$2 \leq a_{n+1} + 3/a_n \leq 6.$$

Divided by 2, the middle term becomes a_{n+2}. It follows that

$$1 \leq a_{n+2} \leq 3.$$

This completes the induction.

3. [30] Let A, B be finite sets who respective cardinalities m, n. Put $C = A \cap B$, and assume it has cardinality k.

(a) In no more than 10 lines, prove that $B - C$ has cardinality $n - k$.

(Hint: $B = C \cup (B - C)$.)

(b) In no more than 10 lines, prove that the $A \cup B$ has cardinality $m + n - k$. (Hint: Write $A \cup B$ as the union of two disjoint sets.)

(a) Recall that (homework) the cardinality of the union of two disjoint nonempty finite sets is the sum of their cardinalities. This is also true when either set is empty. Since C and $B - C$ are disjoint and since B is their union, it follows that $n = k + l$ where k, l are the respective cardinalities of $C, B - C$. Thus $l = n - k$.

(b) We have $B - A = B - C$, since $x \in B - A \Leftrightarrow x \in B$ and $x \notin A \Leftrightarrow x \in B$ and $x \notin A \cap B$. It follows that

$$A \cup B = A \cup (B - A) = A \cup (B - C).$$

Since A and $B - C$ are disjoint, $A \cup B$ has cardinality m plus the cardinality of $B - C$, which is $n - k$ by part (a). This completes the proof.