Math 30b **Homework 3** **Spring, 2009**

Due Wednesday, February 11

1. You will prove another theorem about the polynomial ring $F[x]$ over a field F that mimics a familiar theorem about $\mathbb{Z}$. Recall that by the Unique Factorization Theorem, every nonzero $f \in F[x]$ can be uniquely factorized into the product of a constant and a list of irreducible monic polynomials. Let $f, g \in F[x]$ be nonzero polynomials. Define $gcd(f, g)$ to be the product of all the irreducible monic polynomials appearing in both the factorizations of f and g. (For example, $gcd(x^2 - 1, x + 1) = x + 1$ and $gcd(x^2 - 1, x) = 1$.)

   i. Suppose that $gcd(f, g) = 1$. Use the division algorithm to prove that there exists $a, b \in F[x]$ such that

   $$af + bg = 1.$$

   Conclude the following relation between ideals $(f) + (g) = F[x]$. (Hint: Assume $deg\ f \geq deg\ g$, and do induction on $deg\ g$.)

   ii. Conversely, suppose $(f) + (g) = F[x]$. Prove that $gcd(f, g) = 1$.

   iii. More generally, let $h = gcd(f, g)$. Prove that $(f) + (g) = (h)$. (Hint: Consider $f/h, g/h$.)

   iv. Let $E \geq F$ be an extension field, $\alpha \in E$ a given algebraic element over F, and $f \in F[x]$ be its irreducible polynomial. Suppose $g \in F[x]$ and $g(\alpha) \neq 0$. Prove that $gcd(f, g) = 1$.

   v. Continue the assumptions in iv. Using iv (but without using the proof in class that $F[\alpha]$ is a field) prove that there exists $b \in F[x]$ such that $b(\alpha)g(\alpha) = 1$.

2. Problems 2,11,13, 24b,29,31 Pages 272-273.