**Math 30b** **Homework 4** **Spring, 2009**

Due Wednesday, March 4

1. (a) *A bit of elimination theory.* Let $E \geq F$ be an extension field of $F$, and $\alpha, \beta \in E$ be algebraic over $F$ whose respective irreducible polynomials over $F$ are

$$x^2 + a, \quad x^2 + b,$$

where $a \neq b$. Find the irreducible polynomials of $\alpha + \beta$, $\alpha\beta$, and $\alpha^{-1}$, over $F$, explicitly. Each of your polynomials should be a monic whose coefficients are expressions involving only $a$ and $b$. (Hint: Compute $(\alpha + \beta)^2$ and try to eliminate all appearances of $\alpha$ and $\beta$. You may need to square this again.)

(b) Do this for fun. Do not hand in your solution, but feel free to discuss it with me. Repeat the preceding problem, but the irreducible polynomials of $\alpha, \beta$ are in more general form:

$$x^2 + a_1 x + a_0, \quad x^2 + b_1 x + b_0$$

which are assumed to be distinct.

2. section 31: problems 2,3,26,30,33.

3. (a) (Bonus) Let $E \geq F$ be an extension field of $F$, and $\alpha \in E$ be transcendental over $F$. If $F$ is an uncountable set, prove that the dimension of $F(\alpha)$ over $F$ is uncountable. In other words, there is no finite or countable basis for $F(\alpha)$ in this case.

(b) Do this for fun. Do not hand in your solution, but feel free to discuss it with me. Under the same assumptions as in (a), construct an explicit basis of $F(\alpha)$ over $F$ without using Zorn's Lemma. Explicit here means that your basis must be described in terms of $F$ and $\alpha$. (Hint: Partial fractions.)

4. *Existence of maximal ideal.* Do this for fun. Do not hand in your solution, but feel free to discuss it with me. Let $R$ be a given unital commutative ring. Let $I \neq R$ be an given ideal of $R$. You will prove that every proper ideal of $R$ is contained in a maximal ideal. Let

$$S = \{J | J \text{ is an ideal of } R \text{ s.t. } I \subset J \neq R\}$$

i.e. $S$ is the set of proper ideals of $R$ containing $I$.

a. Verify that the relation on $S$ defined by inclusion $J \subset K$ is a *partial ordering* on $S$. In other words, for any $J, K, L \in S$, we have

i. $J \subset J$,

ii. $J \subset K$ and $K \subset J \Rightarrow J = K$, and

iii. $J \subset K$ and $K \subset L \Rightarrow J \subset L$.

b. Suppose $T \subset S$ is a subset such that $K, L \in T \Rightarrow K \subset L$ *or* $L \subset K$. Verify that the set $J = \cup_{K \in T} K$ is an ideal of $R$ containing $I$ and every $K \in T$. (We say that $J$ is an upper bound of $T$.)

c. Apply Zorn's lemma to conclude that $S$ contains a maximal ideal of $R$.

5. *Existence of algebraic closure (Artin.)* Do this for fun. Do not hand in your solution, but feel free to discuss it with me. You will prove the existence of an algebraic closure of any given field $F$.

a. Let $E_0 \subset E_1 \subset E_2 \subset \cdots$ be a sequence of fields such for each $i$, any non-constant polynomial in $E_i[x]$ has a root in $E_{i+1}$. Put $E = \cup_{i \geq 1} E_i$. Prove that $E$ is field containing every $E_i$ as a subfield.

b. Prove that $E$ it is algebraically closed, i.e. every non-constant polynomial in $E[x]$ has a root in $E$. In particular, $E$ is an algebraic closure of $E_0$ (and of every $E_i$.) Thus, to construct an algebraic closure of $F$, it suffices to find a sequence $F \subset E_1 \subset E_2 \subset \cdots$ of fields satisfying property a. We do this next.

c. For each non-constant polynomial $f \in F[x]$, introduce a symbol $x_f$. Think of the set $S = \{x_f | f \in F[x] \ non-constant\}$ as a set of commuting and distinct variables, one for each non-constant $f \in F[x]$. Let $F[S]$ be the set of all polynomials in those variables with coefficients in $F$. Thus a monomial in $F[S]$ is a word of the form $x_{f_1} \cdots x_{f_k}$ (all the letters commute) where $f_1, .., f_k \in F[x]$ are a finite number of non-constant polynomials. A polynomial in $F[S]$ is defined to be a linear combination of monomials with coefficients in $F$. This makes $F[S]$ a vector space over $F$ with a basis consisting of all monomials. Write down the rule for multiplying two polynomials in $F[S]$, so that $F[S]$ becomes a unital commutative ring.

d. Note that for each $f \in F[x]$, we get $f(x_f) \in F[S]$. For example, if $f = x^2 + x - 1$, then $f(x_f) = x_f^2 + x_f - 1$. Let $I$ be the ideal generated by all $f(x_f)$ in $F[S]$ such that $f \in F[x]$ is non-constant. Thus, $p \in F[S]$ is a member of $I$ iff it is a finite sum of the form

$$p = g_1 f_1(x_{f_1}) + \cdots + g_n f_n(x_{f_n})$$

for some $g_1, .., g_n \in F[S]$ and $f_1, .., f_n \in F[x]$. Now comes the hardest part of this exercise: prove that $I$ is a proper ideal. (Hint: If not, then 1 is equal to an expression like $p$ above. What happens when you substitute $x_{f_i}$ with some root of $f_i \in F[x]$?)

e. Use the preceding problem to conclude that $I$ is contained in a maximal ideal $J$ of $F[S]$. Show that the map $F \to F[S]/J$, $a \mapsto a + J$, is an injective map of fields. Thus, we can regard $F[S]/J$ as an extension field of $J$.

f. Prove that any non-constant polynomial in $F[x]$ has a root in the field $F[S]/J$.

g. Apply a-b to conclude that $F$ has an algebraically closed extension $E$.

h. Show that the field $E'$, consisting of all elements $\alpha \in E$ that are algebraic over $F$ is an algebraic closure of $F$.