Toward a Synthetic Cartan-Kahler Theorem

1 Introduction

The goal of these notes is to build up enough of the foundations and practice of synthetic differential geometry so that we may formulate several important classical theorems and constructions of differential geometry. We hope to illustrate along the way some of the advantages (e.g. cartesian closedness) and pitfalls (e.g. having to use only constructive logic) of the synthetic theory along the way. This endeavor will proceed in several stages.

First, we give a brief overview of the theory of $C^\infty$-rings, used in differential geometry in a way parallel to the functorial approach to algebraic geometry. At this stage we will also discuss some particular classes of ideals of $C^\infty$-rings (namely closed ideals and germ determined ideals) that will be of essential categorical importance later. We quote many results with reference (most often to [?]), but provide proof for very few for the sake of brevity.

Then we will proceed to define a series of categories based off these $C^\infty$-rings, in which the usual category of smooth manifolds $\mathbb{M}$ embeds fully and faithfully. The idea is that $\mathbb{M}$ lacks several desirable features, such as being Cartesian closed, having finite inverse limits in general and having ”infinitesimal” manifolds (which ultimately allow one to circumvent the use of limits in the development of calculus). These categories in which $\mathbb{M}$ embeds enjoy more and more of the properties we think a theory of manifolds should have at each stage of definition. In fact, at a certain stage we fix a Grothendeick topos, $\mathcal{G}$, a category of sheaves. At this point, included in our discussion will be a brief explanation of the topology on the relevant site, the subobject classifiers of the resulting topos and the internal semantics of such a category, which are functorial nature. In particular, we will list some properties of the (internal) geometry of the ”real numbers object” (which is in general not a field) and outline the structure of the synthetic tangent bundle. Here we will discuss the generalized Kock-Lawvere axiom, on which many synthetic proofs hinge. It is such categories that provide the example models of synthetic theory we wish to do geometry in.

With the definition of synthetic tangent bundle in place, we can check that it has the ”usual” hallmarks of the classical notion of tangent bundle (a fiber-wise $R$-module structure) under the assumption of appropriate linear relation to the infinitesimal manifolds that exist in $\mathcal{G}$, infinitesimal linearity. From this point we can define a notion of infinitesimal parallel transport, from which one can build another class of objects, microlinear objects that have a certain exactness property in regard to diagrams of infinitesimally linear objects. Thus, we can finally define symmetric affine connections, sprays and provide a synthetic proof of the Ambrose-Singer-Palais Theorem, drawing a correspondence between these two structures.

Following the discussion of connections, we give a classical statement and proof of the Frobenius theorem. This is done in order to set the stage for a synthetic version of the theorem, using the both the combinatorial differential forms of [?] and differential forms viewed as morphisms with domain infinitesimal cubes as in [?] and [?]. The latter approach invokes the use of the so-called ”amazing” right adjoint to exponentiation. A synthetic proof in the flavor of [?] is outlined.

The final two sections provide some view of how the Cartan-Kähler theorem might be formulated (and possibly proved) synthetically and then some effort is made in interpreting synthetically derived results in the classical case.
2 \( C^\infty \)-rings

The following section summarizes some of the important results we will use later. The development follows almost exactly as in [?], though all of the facts can be found in some form or another in [?] or [?]. A brief overview of many important facts can be found in [?]. Recall that in the functorial approach to algebraic geometry, for any \( k \)-algebra \( \phi : k \to A \) and any map \( p : k^n \to k^m \) given by by \( m \)-polynomials in \( n \) variables, there is a map \( A(p) : A^n \to A^m \) making the following diagram commute:

\[
\begin{array}{c}
k^n \\
\downarrow \phi^n \\
A^n \\
p \\
\phi^m \\
k^m \\
\end{array}
\]

Furthermore, this assignment is functorial and preserves projections: \( A(id) = id \), \( A(p \circ q) = A(p) \circ A(q) \) and \( A(\pi_i) = \pi_i \). The upshot of this is that we treat an \( k \)-algebra as a functor. We define an analogous notion for smooth maps \( \phi : R^n \to R^m \). We call the set \( A(R) \) the underlying set of \( A \).

2.1 Definitions and Properties

**Definition 1.** (1) Let \( C^\infty \) be the category such that \( \text{Ob}(C^\infty) = \{ R^n | n \in \mathbb{Z}_{n\geq 0} \} \) and \( \text{Arr}(C^\infty) = \{ \phi : R^n \to R^m | \phi \text{ smooth} \} \). Then a \( C^\infty \)-ring, \( A \), is a finite product preserving functor on objects given by \( A(R^n) = A(R)^n \).

\[ A : C^\infty \to \text{Set} \]

(2) a finitely generated \( C^\infty \)-ring (f.g.) is a \( C^\infty \)-ring of the form \( C^\infty(R^n)/I \) for some ideal \( I \) and non-negative integer \( n \).

(3) a finitely presented \( C^\infty \)-ring (f.p.) is a \( C^\infty \)-ring of the form \( C^\infty(R^n)/I \) for some finitely generated ideal \( I \) and non-negative integer \( n \).

**Example 1.**

- \( X \subset \mathbb{R} \), then the ring of smooth functions on \( X \), \( C^\infty(X) \), has the structure of a \( C^\infty \)-ring. The \( C^\infty \)-structure is given by composition: a smooth map \( h : \mathbb{R}^n \to \mathbb{R}^m \) is "interpreted" as \( C^\infty(X)(h) : C^\infty(X)^n \to C^\infty(X)^m \) given by composition with \( h : (f_1, \ldots, f_n) \mapsto h \circ (f_1, \ldots, f_n) \).

- The ring \( C^\infty(\mathbb{R}^n) \) is the free \( C^\infty \)-ring on \( n \) generators, these generators being the projections \( \pi_i : \mathbb{R}^n \to \mathbb{R} \).

- The ring of germs of smooth functions at a point \( p \in \mathbb{R}^n \), denoted \( C^\infty_p(\mathbb{R}^n) \), has a \( C^\infty \)-ring structure induced by composition as in the first example. A smooth version of Tietze’s extension theorem tells us that any germ of a function around \( p \) can be extended to a function on all of \( \mathbb{R}^n \). Then \( C^\infty_p(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)/m^9_{\{p\}} \), where \( m^9_{\{p\}} \) is the ideal of functions having zero germ at \( p \).

It is apparent that \( C^\infty \)-rings form a category with (smooth) natural transformations being the morphisms. The underlying set of the \( C^\infty \)-ring \( A \) has the structure of a commutative ring with smooth addition and multiplication. We state some basic properties of \( C^\infty \) rings, most without proof, some with only brief outlines.
Theorem 1. Given a $C^\infty$-ring $A$ and an ideal $I \subset A$, the projection $p : A \to A/I$ induces a $C^\infty$-structure on $A/I$, making $p$ into a $C^\infty$-homomorphism (i.e. a natural transformation).

It is important that we detail how we may realize the standard notion of the ring of $C^\infty$ functions on an open set $U \subset \mathbb{R}^n$ as a $C^\infty$-ring according to our definition. Namely, we will employ the construction of localization at some set of $C^\infty$-functions. In order to do this, we will discuss coproducts of $C^\infty$-rings, and remark about directed and inverse limits of $C^\infty$-rings in general. Given an inverse system of $C^\infty$-rings, $(A_i, \phi_{ij})$, we construct the inverse limit as the inverse limit of the underlying sets of the $A_i$'s:

$$B := \lim_{\leftarrow} A_i \iff B(\mathbb{R}^n) = \lim_{\leftarrow} A_i(\mathbb{R}^n) = \lim_{\leftarrow} A_i(\mathbb{R})^n.$$  

We can make the analogous statement for directed systems of $C^\infty$-rings. Namely, directed limits are computed as directed limits of the underlying sets.

We denote the coproduct of $C^\infty$-rings, $A, B$, by $A \otimes_\infty B$. Immediately from the universal property defining coproduct, we see for ideals $I \subset A, J \subset B$ that we have

$$A/I \otimes_\infty B/J \simeq A \otimes_\infty B/(I, J),$$

where $(I, J)$ is the ideal generated by the images of $I$ and $J$ under the canonical inclusions of the coproduct. Note also that since $C^\infty(\mathbb{R}^n)$ is free on $n$ generators we have

$$C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m) \simeq C^\infty(\mathbb{R}^n \times \mathbb{R}^m),$$

where the coproduct inclusions come from the projections from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Lemma 1. Every open set $U \subset \mathbb{R}^n$ has a smooth approximation of a characteristic function, $a : \mathbb{R}^n \to \mathbb{R}$ such that $U = a^{-1}(\mathbb{R} \setminus \{0\})$.

Via such a function, we see that for $U \subset \mathbb{R}^n$, there is a diffeomorphic closed subspace $U^* \subset \mathbb{R}^{n+1}$ given by $U^* = \{(x, y) \mid y \cdot a(x) = 1\}$.

We may construct localization of $C^\infty$-rings as follows: restricting to the case of inverting one function, $f$, let $A$ be a $C^\infty$-ring. We may write $A$ as the colimit of finitely generated $C^\infty$-rings $\{A_i\}$ and localize at each of those. So it suffices to consider $A$ as finitely generated. If $\eta : A \to A\{f^{-1}\}$ is the monomorphism that gives the localization $A\{f^{-1}\}$ its universal property, then for $A$ finitely generated we have $(C^\infty(\mathbb{R}^n)/I)\{f^{-1}\} \simeq C^\infty(\mathbb{R}^n)/(\eta(I))$. In particular, using partitions of unity we can show the following:

Proposition 1. $a \in C^\infty(\mathbb{R}^n)$ and $U = a^{-1}(\mathbb{R} \setminus 0)$. Then

$$C^\infty(\mathbb{R}^n)\{a^{-1}\} \simeq C^\infty(U).$$

Furthermore,

$$C^\infty(U) \simeq C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1).$$

With the observation that any $C^\infty$-ring is a directed colimit of finitely generated $C^\infty$-rings we are able to compute any colimit of $C^\infty$-rings.

Example 2. A ring of germs $C_p^{\infty}(\mathbb{R}^n)$ is isomorphic to $\varinjlim_{p \in U} C^\infty(U)$ with maps $C^\infty(U) \to C^\infty(V)$ whenever $V \subset U$.

This matches what we would expect from the analogy with algebraic geometry we are trying to draw.
Theorem 2. $C_p^\infty(\mathbb{R}^n) \otimes_C C_q^\infty(\mathbb{R}^m) \simeq C_{(p,q)}^\infty(\mathbb{R}^{n+m})$, for any points $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$.

Proof: By the statement about quotients and the universal property of coproduct above, we need to show $(m_p^q, m_q^p) = m_{(p,q)}^q$. We show only the containment “⊂”, as the other obvious. Take some function $f : \mathbb{R}^{n+m} \to \mathbb{R}$ such that for some open sets $p \in U$ and $q \in V$ we have $f|_{U \times V} = 0$. Consider the characteristic functions $\chi_U$ and $\chi_V$ of $U$ and $V$, respectively. Then the product $\chi_U \cdot \chi_V \cdot f = 0$ in $C^\infty(\mathbb{R}^{n+m})$. Since $\chi_U$ and $\chi_V$ are invertible in $C^\infty(\mathbb{R}^{n+m})/(m_p^q, m_q^p)$, $f = 0 \in C^\infty(\mathbb{R}^{n+m})/(m_p^q, m_q^p)$, which is to say $f \in (m_p^q, m_q^p)$.

Now we introduce a class of $C^\infty$-rings that will ultimately serve as the modeling objects for “infinitesimal manifolds”.

Definition 2. A Weil algebra, $W$, is a local $\mathbb{R}$-algebra that is finite dimensional as an $\mathbb{R}$-vector space and can be written as a direct sum $W = \mathbb{R} \oplus m$, where $m$ is a maximal ideal of $W$.

Weil algebras will be important to us, because in our early attempts to embed the category of smooth manifolds in a “better behaved” category they will correspond to the first examples of infinitesimal spaces and be the first class of objects for which exponentials are defined. Additionally, they correspond to an important notion in classical differential geometry.

Definition 3. Let $m \subset C_0^\infty(\mathbb{R}^n)$, be the maximal ideal. Then $J_n^k = C_0^\infty(\mathbb{R}^n)/m^{k+1}$ is called the ring of $(k)$-jets.

Rings of jets somehow play the role of the universal objects necessary for the foundations of the Cartan-Kahler theorem. This will be explained near the end of the paper. However, for now we will simply quote some results from chapter 2 of [?] drawing the connection rings of jets and Weil algebras.

Theorem 3. A ring of jets is a Weil algebra.

Proof: Let $J_n^k = C_0^\infty(\mathbb{R}^n)/m^{k+1}$, where $m = \{ f \in C_0^\infty(\mathbb{R}^n) | f(0) = 0 \}$ is the maximal ideal of $C_0^\infty(\mathbb{R}^n)$. By Hadamard’s lemma, we see for $f \in m$,

$$f(x) = \sum_{i=1}^n x_i g_i(x)$$

for $g_i(x) \in C_0^\infty(\mathbb{R}^n)$, and so $m = (x_1, \ldots, x_n)$, the smooth projection maps. So $C_0^\infty(\mathbb{R}^n) = \mathbb{R} \oplus m$. We must show $J_n^k$ is finite dimensional as a vector space. Applying Hadamard’s lemma $k$ times to any $f \in m$ and killing all terms in $m^{k+1}$, we see the constant function 1 and the monomials of homogeneous degree less than or equal to $k$ generate $J_n^k$.

Example 3. One of the simplest examples of both a ring of jets an a Weil algebra is the ring of dual numbers: $\mathbb{R}[\epsilon] = \mathbb{R} \oplus \mathbb{R} \epsilon \simeq C_0^\infty(\mathbb{R})/(\epsilon^2)$.

Corollary 1. $W = \mathbb{R} \oplus m$ a Weil algebra. Then $m^k = 0$ for some $k$.

Proof: This follows from the finite dimensionality of $W$ and Nakayama’s lemma applied to the situation of $m^k = m^{k+1}$.

Informally, we may think of a $k$-jet as the $k$th order truncation of the Taylor expansion around 0 of a smooth function on $\mathbb{R}^n$. Although not all Weil algebras are rings of jets, in some sense rings of jets characterize all Weil algebras.
Theorem 4. (Characterization of Weil algebras) Let $A$ be an $R$-algebra. Then the following are equivalent:

1. $A$ is a Weil algebra.
2. $A$ is isomorphic to an $R$-algebra of the form $R[X_1, \ldots, X_n]/I$ for some ideal $I$ such that for some $k$, $I$ contains all homogeneous monomials of degree $k$.
3. $A$ is isomorphic to $R[[X_1, \ldots, X_n]]/I$ for some ideal $I$ containing $m^k$ for some $k$.
4. $A$ is a quotient ring of jets $J^k_n$.
5. $A$ is isomorphic to a ring $C_\infty^0(R^n)/I$ which is finite dimensional as an $R$-vector space.

From this theorem we have several important corollaries.

Corollary 2. Let $W, W'$ be Weil algebras and $A$ an arbitrary $C_\infty$-ring

- $W$ is a f.p. $C_\infty$-ring.
- Any $R$-algebra homomorphism $W \to A$ or $A \to W$ is a $C_\infty$-homomorphism.
- $A \subset W$ implies $A$ is a Weil algebra.
- $W \otimes_\infty W'$ is a Weil Algebra.

2.2 Classes of Ideals

Once we begin to define the categories that support various models of synthetic reasoning the following definitions will play a crucial role.

Definition 4. Let $A$ be a $C_\infty$-ring.

- $A$ is near-point determined if it can be embedded into a direct product of Weil algebras.
- $A$ is closed if it can be embedded into a direct product of formal algebras, i.e. $C_\infty$-rings of the form $R[[X_1, \ldots, X_n]]/I$
- $A$ is germ determined if it can be embedded into a direct product of pointed local rings, i.e. local $C_\infty$-rings $A_i$ endowed with a $C_\infty$ homomorphisms $A_i \to R$.

Theorem 5. Let $A$ be a $C_\infty$-ring of the form $C_\infty(M)/I$ where $M \in Ob(M)$ (we may even assume $M \subset R^p$). Also, let $Z(I) = \cap\{f^{-1}(0)\} = f \in I$ be the "zero locus" of the ideal $I$. Then

1. $A$ is near-point determined if and only if for all $f \in C_\infty(M)$ we have:

$$\forall x \in Z(I) \text{ and } \forall k \in \mathbb{N}, \text{tay}_x(f) \in (\text{tay}_x(I) + m^k) \Rightarrow f \in I,$$

where $m$ is the maximal ideal of $R[[X_1, \ldots, X_n]]$, $\dim(M) = n$, and $\text{tay}_x$ is the Taylor series around $x$ using local coordinates.

2. $A$ is closed if and only if for all $f \in C_\infty(M)$ we have:

$$\forall x \in Z(I), \text{tay}_x(f) \in \text{tay}_x(I) \Rightarrow f \in I$$
3. A is germ determined if and only if:

\[ \forall x \in Z(I), f|_x \in I|_x \Rightarrow f \in I, \]

Proof of 3: (\(\Rightarrow\)) Take \(f \in A\) nonzero. By definition, we can embed \(A\) into a product of pointed local rings, which means there is a map \(\phi : A \to B\) with \(B\) a pointed local ring such that \(\phi(f) \neq 0\). \(A\) being finitely generated means that \(B\) is as well. A fact which we have not proved but will use is that a f.g. pointed local ring isomorphic to a quotient of a ring of germs: \(B \simeq C^\infty_0(\mathbb{R}^n)/J\), for some ideal \(J\). With the first map \(\phi\) and the second evaluation at 0, say the composition

\[ C^\infty(M)/I \to C^\infty_0(\mathbb{R}^n)/J \to \mathbb{R} \]

corresponds to the point \(p \in Z(I)\). The we can show that \(\phi\) factors through the ring \(C^\infty_p(M)/I|_p\). Thus, \(\phi(f) \neq 0 \to f|_p = 0\).

(\(\Leftarrow\)) The map

\[ A \to \Pi_{x \in Z(I)} C^\infty_x(M), f \mapsto \{f|_x\}_x \]

is readily seen to be injective. \(\square\)

An application of Nakayama’s lemma shows that an f.g. \(C^\infty\)-ring is near point determined if and only if it is closed. It is also worth noting that for \(I\) an ideal, closed implies germ determined. When we begin to construct models for the synthetic viewpoint, we will ultimately be concerned with only germ determined \(C^\infty\)-rings (and make only a remark or two about the so-called ”topos of closed ideals”. We also refer to the ideals by which we quotient to get, for example, germ determined \(C^\infty\)-rings as germ determined ideals.

Proposition 2. If \(I \subset C^\infty(M)\) is germ determined, then for any \(h \in C^\infty(M)\) the ideal \((I, h)\) is germ determined. In particular, any finitely generated ideal is germ determined.

2.3 Manifolds as \(C^\infty\)-rings

Theorem 6. For every smooth manifold \(M\), \(C^\infty(M)\) is finitely presented.

This follows from the \(\epsilon\)-neighborhood theorem, the fact that \(C^\infty(U)\) is finitely presented for an open set \(U \subset \mathbb{R}^n\) and retracts of finitely presented \(C^\infty\)-rings are finitely presented.

Lemma 2. Let \(U \subset \mathbb{R}^n\) and \(V \subset \mathbb{R}^m\) be open. Then

\[ C^\infty(U) \otimes C^\infty(V) \simeq C^\infty(U \times V). \]

By a similar method to the proof of theorem 6, we can use this lemma to show

Theorem 7. Let \(M_1\) and \(M_2\) be manifolds. Then

\[ C^\infty(M_1) \otimes C^\infty(M_2) \simeq C^\infty(M_1 \times M_2). \]

Recall that for a smooth map \(f : M \to N\) and a submanifold \(Z \subset N\), if \(f\) is transversal to \(Z\), we know that \(f^{-1}(Z) \subset M\) is a submanifold. Furthermore, we obtain a pullback diagram in the category of smooth manifolds with vertex \(f^{-1}(Z)\) by pulling \(f\) back along the inclusion \(Z \to N\).

Theorem 8. Let \(f : M \to N\) be transversal to a submanifold \(Z \subset N\). Then the pullback diagram described above is taken to a pushout diagram of \(C^\infty\)-rings.
Theorem 10. The functor componentwise, φ functions for any finite set \( \{a_1, \ldots, a_n\} \), we let \( W_{a_1 \ldots a_n} = W_{a_1} \cap \cdots \cap W_{a_n} \) and
\[
B_{a_1 \ldots a_n} = C^\infty(W_{a_1 \ldots a_n})/(f_1|_{W_{a_1 \ldots a_n}}, \ldots, f_p|_{W_{a_1 \ldots a_n}}).
\]

The limit is defined by the obvious restrictions.

Here we begin to see that it may be possible to preserve at least some of the structure of the category of smooth manifolds we wish to use via some embedding into a larger category with a "broader", or more permissive notion of manifold. Namely, we have the following:

**Theorem 9.** The contravariant functor \( \mathbb{M} \to (C^\infty\text{-rings}) \), sending a smooth manifold \( M \) to \( C^\infty(M) \) is full and faithful, maps into finitely presented \( C^\infty\text{-rings} \), and sends transversal pullbacks to pushouts.

3. The "right" categories

Now we go about defining the categories in which we can fully and faithfully embed \( \mathbb{M} \) and replicate the many of the constructions of classical differential geometry. Our categorical target is a category of sheaves over the formal dual of the category of \( C^\infty\text{-rings} \).

3.1 \( \mathbb{L} \): the category of \( C^\infty \) loci

**Definition 5.** The category of formal \( C^\infty\text{-varieties} \) or \( C^\infty \) loci, denoted \( \mathbb{L} \), is the opposite category of the category of finitely generated \( C^\infty\text{-rings} \).

Specifically, we denote the formal dual of an f.g. \( C^\infty \)-ring \( A \) by \( lA \) and the morphisms are described as follows: \( lB \to lA \), where \( A = C^\infty(\mathbb{R}^n)/I \) and \( B = C^\infty(\mathbb{R}^m)/J \), is an equivalence class of smooth functions \( \phi : \mathbb{R}^m \to \mathbb{R}^n \) with the property that \( f \in I \) implies the \( f \circ \phi \in J \). Here \( \phi \) is equivalent to \( \phi' \) if componentwise, \( \phi_i - \phi'_i \in J \) for \( 1 \leq i \leq m \).

Being the formal dual of the category of \( C^\infty \)-rings, we automatically get the dual of theorem 9:

**Theorem 10.** The functor \( s : \mathbb{M} \to \mathbb{L} \) defined by \( M \mapsto lC^\infty(M) \) is full, faithful and preserves transversal pullbacks.

This is a step in the right direction for our purposes, because now we have covariantly embedded the category of smooth manifolds into the category \( \mathbb{L} \), which underlies all future constructions we will make. In addition to this, we now have some function spaces. Intuitively, when \( lB \) is "sufficiently small", we may form the the function space (or exponential object) maps from \( lB \) to any other \( lA : lA lB \). Let \( R \) denote \( lC^\infty(\mathbb{R}) \). Let us define some important infinitesimal loci, which in particular are subloci of \( R \) at 0.

- The k-th order infinitesimals are defined as \( D_k = l(C^\infty(\mathbb{R})/(x^{k+1})) = \{ x \in R | x^{k+1} = 0 \} \). In particular, we write \( D = l(C^\infty(\mathbb{R})/(x^2)) = \{ x \in R | x^2 = 0 \} \).
- In n dimensions, we define \( D_k(n) = l(C^\infty(\mathbb{R}^n)/m^{k+1}) = \{ x \in R^n | x^a = 0, \forall a, |a| = k+1 \} \), where \( a = (a_1, \ldots, a_n) \) and \( |a| = a_1 + \cdots + a_n \).
- If \( p \in \mathbb{R}^n \), we define \( \Delta_p = lC_p^\infty(\mathbb{R}^n) \).
For all \( k \) and \( n \), we have \( D_k(n) \subset \Delta_0 \).

Notice that the \( C^\infty \)-rings dual of the infinitesimal loci we have defined are all Weil algebras according to our earlier characterization. For an arbitrary locus, \( l(C^\infty(\mathbb{R}^n)/I) = lA \), and \( p \in \mathbb{R}^n \), the intersection \( lA \cap \Delta_p \) is the germ of \( lA \) at \( p \). This constitutes an example of the notion of "very small" manifold, a topic we will return to shortly. However, for now let us discuss the analogue of the tangent bundle in the category \( \mathbb{L} \).

**Definition 6.** For an arbitrary locus \( l(C^\infty(\mathbb{R}^n)/I) = lA \), we define the tangent locus as

\[
T(lA) = l\left( C^\infty(\mathbb{R}^n \times \mathbb{R}^n)/(I(x), \{ y_i \frac{\partial f}{\partial x_i} \}) \right),
\]

where \( x = (x_1, \ldots, x_n) \) are the coordinates of the first copy of \( \mathbb{R}^n \) and \( y = (y_1, \ldots, y_n) \) are the coordinates of the second copy.

This notion of tangent locus can be expressed as an exponential, and in the case of a manifold corresponds to the usual notion of tangent bundle.

**Theorem 11.**
1. There is a natural 1-1 correspondence of morphisms in \( \mathbb{L} \) of the form \( lC \to T(lA) \) and of the form \( lC \times D \to lA \) for arbitrary loci \( lA, lC \). That is to say, \( lA^D \simeq T(lA) \).

2. For \( M \in \mathbb{M} \) Then the functor \( s \) preserves the tangent bundle of \( M \). Namely, \( s(M)^D \simeq s(TM) \) and the projection map \( \pi : TM \to M \) in \( \mathbb{M} \) is mapped to \( s(\pi) = ev_0 : s(M)^D \to s(M) \), evaluation at \( 0 \in D \).

This means that in \( \mathbb{L} \), we think of a tangent vector to \( s(M) \) at a point \( p \) as a map \( f : D \to s(M) \) such that \( f(0) = p \). We now point out that all duals of Weil algebras are always exponentiable.

**Theorem 12.** Let \( W \) be a Weil algebra. Then for any \( lA \in \mathbb{L} \) the exponential \( lA^{lW} \) exists in \( \mathbb{L} \).

**Proof Sketch:** With \( A = C^\infty(\mathbb{R}^n)/I \), we realize \( lA \) as the joint coequalizer of \( f : R^n \to R \), for all \( f \in I \) and the zero map \( 0 : R^n \to R \). Exponentiation preserves all inverse limits, and this means we need only check that \( (R^n)^{lW} \) exists in \( \mathbb{L} \), which amounts to show \( R^{lW} \) exists. To do this we take any ideal \( J \) consider the locus \( lB = l(C^\infty(\mathbb{R}^m)/J) \) and suppose \( W = C^\infty(\mathbb{R}^m)/(m^k + J) \). Then we see the elements of \( B \otimes_{\mathbb{L}} W \) as smooth functions of the correct form. \( \square \)

There are a number of important coreflective subcategories of \( \mathbb{L} \), one of which we will consider extensively. Recall that a subcategory \( D \subset C \) being coreflective implies the inclusion functor admits a right adjoint: \( r : C \to D \), called the reflector of \( D \).

Consider the opposite category of the category of germ determined \( C^\infty \)-rings, \( G \), a full reflective subcategory of \( \mathbb{L} \). The inclusion \( G \hookrightarrow \mathbb{L} \) has adjoint, \( \lambda : \mathbb{L} \to G \) defined by \( l(C^\infty(\mathbb{R}^n)/I) \mapsto l(C^\infty(\mathbb{R}^n)/I^\sim) \). Here \( I^\sim \) the smallest germ determined ideal containing \( I \). Formally, we say

\[
f \in I^\sim \iff \forall x \in Z(I), f|_x \in I|_x,
\]

where \( Z(I) \) is the formal zero set of the ideal \( I \). After passing to the category of smooth functors we will restrict our attention to the category of sheaves on \( G \) in order to obtain ”nicer” properties of the object (sheaf, or variable set) which plays the role of the real line. For now we simply note

**Theorem 13.** Let \( W \) be a Weil algebra. Then \( lW \) is exponentiable in \( G \).

This follows from the previous theorem, a germ determined ring being embedded in a product of pointed local \( C^\infty \)-rings, the product of Weil algebras is a Weil algebra and using the reflector \( \lambda \) of \( G \).

**Remark 1.** remark 1.14 on prolongations
3.2 $\text{Set}^{L\text{op}}$: the category of smooth functors

In defining $L$ we have seen the category of smooth manifolds embeds covariantly and has "good exactness properties." However, we want more: we would like all objects of our ambient category, including objects of $s(M)$, to be exponentiable. Furthermore, we would like to endow our real line object (in the category $L$, for example, we had $R = s(\mathbb{R})$ play the role of a model for manifolds, just as $\mathbb{R}$ does in $M$) with as many properties as possible that $R \in M$ enjoys. In order to achieve this in a structure preserving way, we embed $L$ in the obvious category $\text{Set}^{L\text{op}}$, the category of presheaves over $L$, or so-called smooth functors.

$L$ embeds in $\text{Set}^{L\text{op}}$ via the **Yoneda embedding**: $Y : L \to \text{Set}^{L\text{op}}$ defined by $Y(lA) = \text{Hom}_{L}(\cdot, lA)$. We will usually write $lA$ for $Y(lA)$ for brevity. $Y$ preserves inverse limits and exponentials (whenever they exist in $L$), so by composing with our embedding $s : M \to L$ we have an embedding of smooth manifolds into the category of smooth functors, $Y \circ s = s'$.

The category of smooth functors, like any presheaf category, can be thought of as having "variable sets" as its objects. Such a variable set is parameterized by smooth loci. Notice that in addition to the embedding of the category $M$ in $\text{Set}^{L\text{op}}$, the category $\text{Set}$ also embeds in $\text{Set}^{L\text{op}}$. The functor $c : \text{Set} \to \text{Set}^{L\text{op}}$ is defined by $c(S)(lA) = S$ for all $lA \in L$. $c$ has a right adjoint, the **functor of global sections**, $\Gamma : \text{Set}^{L\text{op}} \to \text{Set}$ defined by $\Gamma(F) = F(1)$, where $1$ is the point: $1 = s(\{\} \ ) = l(\text{C}^{\infty}(\mathbb{R})/(x))$. For example, in the case $F$ is representable by $lB$, $\Gamma(F) = \text{Hom}_{L}(l(\text{C}^{\infty}(\mathbb{R})/(x)), lB)$.

In general, for any small category $C$, the category of presheaves over $C$ is a topos, meaning it has finite limits, is cartesian closed and has power objects. We will forego the proofs of these facts, which may be found in [?] or [?], and instead exhibit explicitly the subobject classifier $T : 1 \to \Omega$ of the category $\text{Set}^{L\text{op}}$. All of the important topos structure hinges on this construction. Our discussion is based on [?].

**Definition 7.** Let $C$ be a cartesian closed category with finite limits. A **subobject classifier** is an object $\Omega$ together with a map, called **true**, $T : 1 \to \Omega$, such that for any monomorphism $i : X \to Y$ there is a unique arrow $\chi_{i} : Y \to \Omega$ making the square below a pullback diagram.

\[
\begin{array}{ccc}
X & \to & 1 \\
\downarrow i & & \downarrow T \\
Y & \to & \Omega \\
\end{array}
\]

The idea is that $\chi_{i}$ is a "characteristic morphism for the subobject $X$ of $Y$. This extends the notion of characteristic function from the category of sets and functions. In dealing with topoi it is always useful to keep in mind that a topos is category in which the objects behave like sets, or put another way, a generalized universe of sets. Slightly more precisely, any construction involving usual sets will be able to be carried out in a topos. We will make this more precise in this section and the next.

**Example 4.** In the category $\text{Set}$, $1 = \{\ast\}$ is the one point set, $\Omega = \{0, 1\}$ and the true arrow is defined by $T(\ast) = 1$. One can easily verify that for sets $A \subset B$, the characteristic function of $A$ in $B$ makes the relevant square a pullback.

In order to determine the subobject classifier in $\text{Set}^{L\text{op}}$ we must define the notion of a sieve, which is also of importance in defining the categorical notion of (Grothendieck) topology, a concept which bares fundamental resemblance to the classical notion of topology and simultaneously diverges enough from the original concept to cover a distinct territory of formalism.
Definition 8. Let \( C \) be a category. A sieve \( S \) over \( X \in C \) is a subfunctor of \( \text{Hom}_C(-, X) \). That is to say, \( S \) is a collection of arrows of \( C \) with common codomain \( X \) which is absorptive with respect to composition.

The subobject classifier of \( \text{Set}^{\text{op}} \): Consider the functor given on objects by \( \Omega(lA) = \text{the set of all sieves over } lA \) and on morphisms \( \phi : lA \rightarrow lB \), \( \Omega(\phi) \) is given by precomposition with \( \phi \). The terminal object in \( \text{Set}^{\text{op}} \) is the constant presheaf with value the one element set, \( c(\{\ast\}) = 1_{\text{Set}^{\text{op}}} \), and the natural transformation true \( T : 1_{\text{Set}^{\text{op}}} \rightarrow \Omega \) picks out the maximal sieve for each object \( lA \), i.e. the sieve of all morphisms with a fixed codomain \( lA \).

To give some indication of how this is a subobject classifier of \( \text{Set}^{\text{op}} \), take \( F \) a presheaf over \( \mathbb{L} \) with sub-presheaf \( G \subset F \). Then define the natural transformation \( \chi_F : F \rightarrow \Omega \) for an arbitrary object \( lA \in \mathbb{L} \) as \( (\chi_F)_lA : F(lA) \rightarrow \Omega(lA), f \mapsto \{ \phi : lB \rightarrow lA | \phi^*(f) \in G(lB) \} \) where \( \phi^* \) is the induced map of sheaves by \( \phi \). One can readily check that this makes the relevant square a pullback.

It is at this point we must depart from classical logic. For a general topos, the subobject classifier has the structure of an internal Heyting algebra, meaning using arrows of the topos we may express the facts that \( \Omega \) is a lattice which when viewed as a poset admits a universal implication operation [come back and flesh this out]. An (internal) Heyting algebra is Boolean if the law of the excluded middle holds, and we call a topos Boolean if its subobject classifier is an internal Boolean algebra. It should be noted here that due to the Heyting algebra structure of \( \Omega \) in a topos \( \mathcal{E} \), that given any object \( A \) of \( \mathcal{E} \) the subobjects of \( A \), \( \text{Sub}(A) \), themselves form a Heyting algebra. A topos ”supports” classical logic precisely when \( \text{Sub}(A) \) Boolean for all objects \( A \) by proposition VI.I.1 of [?].

We aim to show that the presheaf \( \Omega \) defined above cannot be a Boolean algebra. Namely, we want to exhibit a smooth functor \( lA \) that has Heyting algebra \( \text{Sub}(lA) \) that is not Boolean.

Elements at stages of definition, semantics

Theorem 14 (2.4). The following statements are valid in \( \text{Set}^{\text{op}} \)

why \( R \) sucks here: lack of being a local ring, nonarchemedian, compactness properties, generalized Kock-Lawvere axiom

3.3 \( \mathcal{G} = \text{Set}^{\text{op}} \) and the Topology of Manifolds in \( \mathcal{G} \)

We now take a slight detour into topos theory in order to fix the choice of model we will work with in our further investigations into the theory of manifolds.

Definition 9. Let \( C \) be a small category. A Grothendieck topology, \( J \) on \( C \) is given by specifying for each object \( U \) of \( C \) a collection of sieves \( J(U) \) satisfying the following:

1. \( \forall U \in \text{ob}(C), \bigcup_{A \in \text{ob}(C)} \text{Hom}(A, U) \in J(U) \)

2. If \( R \in J(U) \) and \( f : V \rightarrow U \) is a morphism of \( C \), then \( f^*(R) = \{ \alpha : W \rightarrow V | f \circ \alpha \in R \} \in J(V) \)

3. If \( R \in J(U) \) and \( S \) is a sieve over \( U \) not necessarily in \( J(U) \) such that for each \( (f : V \rightarrow U) \in R \) we have \( f^*(S) \in J(V) \), then in fact \( S \in J(U) \).

Definition 10. A collection of arrows in the category \( \mathcal{G} \) with common codomain \( \{ lA_\alpha \xrightarrow{i_\alpha} lA \} \) is called a covering family of \( lA \) if

1. \( \forall \alpha \) there is a \( b_\alpha \) such that \( lA_\alpha \xrightarrow{i_\alpha} \lambda l(A\{b_\alpha^{-1}\}) \xrightarrow{g_\alpha} lA \) is the same morphism as \( f_\alpha \), where \( g_\alpha \) is the image of the canonical inclusion of \( A \) into a localization
2. Suppose $A_\alpha = C^\infty(\mathbb{R}^n)/I_\alpha$ and $A = C^\infty(\mathbb{R}^n)/I$. Then for $x \in Z(I)$ there is an $\alpha$ such that $x = f^*_\alpha(y)$ where $y \in Z(I_\alpha)$ and $f^*_\alpha$ is the induced map on formal zero loci of $f_\alpha$.

Lemma 3. The covering families described above form a Grothendieck topology, $T$, on $\mathbb{G}$.

Proposition 3. The site $(\mathbb{G}, T)$ is subcanonical, i.e. all the representable functors $\mathbb{G}(\cdot, lB)$ are sheaves.

Using this topology we form the topos $\mathbb{G} \approx Sh(\mathbb{G}^{op}, T)$. This is the topos of presheaves which satisfy the sheaf axiom with respect to the topology $T$.

Proposition 4. $\mathbb{G}$ is a topos and the following are valid:

1. $R$ is a unital commutative ring
2. $R$ has order relations $<$ and $\leq$ that are compatible with the ring structure
3. $R$ is a local ring
4. $R$ is an Archimedean ring, i.e. $\mathbb{G} \models \forall x \in R$ there is an $n \in \mathbb{N}$, $x < n$.

Theorem 15. The generalized Kock-Lawvere axiom holds in $\mathbb{G}$: any element of the exponential object $R^{D_{\alpha}(n)}$ is given by a unique polynomial in $n$ variables of total degree $k$.

We describe (internal) open covers from $\mathcal{G}$ in order to make a more detailed account of the theory on manifolds in $\mathbb{G}$. We should also note that, just as in $Set^{L^{op}}$, we must reason intuitionistically in $\mathbb{G}$, and the objects can be simultaneous thought of as variable (and in this case continuously variable) sets and spaces.

Remark 2. We can recreate the construction of $\mathbb{G}$, but instead of using $\mathbb{G}$ we consider the full subcategory of $L$ consisting of duals of closed $C^\infty$-rings, denoted $F$. We can see right away that $\mathbb{G} \hookrightarrow \mathbb{F} \hookrightarrow L$, and it is fact that $\mathbb{F}$ is a coreflective subcategory of $L$. So we may make the construction $\mathcal{F} := Set^{F^{op}}$, just as in the case for $\mathbb{G}$. Both $\mathcal{G}$ and $\mathcal{F}$ are so-called (by [?]) well adapted topos models, specifically, topoi for which the synthetic framework of differential geometry can be realized. The category $\mathcal{F}$ is different from $\mathbb{G}$ most prominently in the treatment of the ordering on $R$ and the integration axiom. We will not pursue these differences. For the remainder of these notes we will work in the category $\mathbb{G}$.

4 Connections

We now begin to develop the machinery to discuss connections in the synthetic context. Again, we fix the topos $\mathbb{G}$ and work exclusively inside it.

4.1 Infinitesimal Parallel Transport

As given as an exercise in [?], we can use the inclusion $D(2) \subset D \times D$ to construct a natural map

$$T(TM) \rightarrow TM \times_M TM,$$

or phrased in a more categorical way and using the adjunction between product and exponentiation afforded by cartesian closedness

$$(M^D)^D \simeq M^{D \times D} \rightarrow M^D \times_M M^D$$

under certain conditions on $M$. Namely, $M$ must be infinitesimally linear. Recall that $D(n) = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_i x_j = 0 \forall i, j \}$ and $inc_i : D \rightarrow D(n)$ is given by $d \mapsto (0, \ldots, d, \ldots, 0)$. Also, note that $M^{D(n)}$ may be thought of as the polynomials in $n$ variables of total degree 1 by the generalized KL-axiom.
Definition 11. An object $M$ is called infinitesimally linear if for all positive integers $n$ and each $n$-tuple of maps $t_i : D \to M$ with $t_1(0) = \cdots = t_n(0)$, there exists a unique $p : D(n) \to M$ with $p \circ \text{inc}_i = t_i$

To construct our natural map, we must show that $TM \simeq M^D \xrightarrow{\pi} M$, $\pi(t) = t(0)$, has an $R$-module bundle structure. That is to say, for $x \in M$ we want an $R$-module structure on $T_x M = (M^D)_x$.

Definition 12. A tangent vector to $M$ at $x \in M$ is a map $t : D \to M$ such that $t(0) = x$.

Lemma 4. If $M$ is infinitesimally linear, the $(M^D)_x$ has an $R$-module structure.

Proof Sketch: By infinitesimal linearity, given $t_1, t_2 \in (M^D)_x$, there exists a unique total degree 1 polynomial in two variables $p \in D(2) \to M$ with $p \circ \text{inc}_1 = t_1$ and $p \circ \text{inc}_2 = t_2$. Then set $t_1 + t_2 := p \circ \Delta$, where $\Delta : D \to D(2)$ is the diagonal map $\Delta(d) = (d, d)$. Then the axioms of an $R$-module action follow from the KL-axiom with this definition.

Now by cartesian closedness, we know there is a map $M^i : M^{D \times D} \to M^{D(2)}$ (the image of the inclusion map $i : D(2) \to D \times D$). This map is just given by restricting $t \in M^{D \times D}$ to $D(2)$. Furthermore, by infinitesimal linearity of $M$ we know $M^D \times_M M^D \simeq M^{D(2)}$ (as in [?], I.6). It is not hard to check that the map $M^i$ is compatible with the fiber-wise $R$-module structures present, and so this is our natural map.

In the terminology of [?], a section of $M^i$ gives some notion of an infinitesimal parallel transport. We return to the reference [?] to relate this concept to that of an affine connection.

4.2 Affine Connections

There is an additional restriction on spaces/objects $M$ (which the embedding of all classical manifolds satisfies) needed to make precise the notion of affine connection that deals with somehow respecting the structure of the exponentiation functor $R^{(-)}$.

Definition 13. 1. For any object $M$, let $\mathcal{D}$ be a finite co-cone of infinitesimally linear objects. $\mathcal{D}$ is called an $M$-colimit if the functor $M^{(-)}$ sends $\mathcal{D}$ into a limit diagram.

2. An object $M$ is said to be a microlinear if every $R$-colimit is an $M$-colimit.

We record some elementary facts about the collection of microlinear objects.

Proposition 5. 1. $R$ is trivially microlinear.

2. Any inverse limit of microlinear objects $s$ is again microlinear.

3. If $M$ is a microlinear object and $X$ is an arbitrary object, then $M^X$ is microlinear.

4. The tangent bundle of a microlinear object has a fiber-wise $R$-module structure that is functorial in $M$.

Due to the dependence on $R$ in the definition of microlinear we have the following:

Theorem 16. The canonical map $\alpha : (M^D)_x \times (M^D)_x \to (M^D)_x \xrightarrow{\alpha} \Delta$ given by $\alpha(t, s)(d) = t + d \cdot s$ is a bijection for each $x \in M$.

Definition 14. For $M$ microlinear, an symmetric affine connection on $M$ is a map $\nabla : M^{D(2)} \to M^{D \times D}$, that is a section of $M^i$ and satisfies the following conditions:
\[ \nabla(t_1, t_2)(d_1, 0) = t_1(d_1), \quad \nabla(t_1, t_2)(0, d_2) = t_2(d_2) \]

2. \[ \nabla(at_1, t_2)(d_1, d_2) = \nabla(t_1, t_2)(ad_1, d_2) \quad \text{and} \quad \nabla(t_1, at_2)(d_1, d_2) = \nabla(t_1, t_2)(d_1, ad_2) \]

3. \[ \nabla(t_1, t_2)(d_1, d_2) = \nabla(t_2, t_1)(d_2, d_1). \]

In our setting (namely, when \( M \) is microlinear), affine connections correspond to infinitesimal parallel transports.

**Theorem 17.** \( H : V \to W \) a map of \( \mathbb{R} \) modules and \( W \) microlinear satisfying the KL-axiom. Then \( H \) is linear \( \iff \) \( H \) is homogeneous.

**Definition 15.** A spray is a map \( \sigma : M^D \to M^{D_2} \) that is fiber-wise a homogeneous section of the natural restriction map \( M^I : M^{D_2} \to M^D \) induced by the inclusion \( i : D \to D_2 = \{ d \in \mathbb{R} | d^3 = 0 \} \).

**Theorem 18.** (Amborse, Singer, Palais) \( M \) microlinear, Then there is a natural 1-1 correspondence between symmetric affine connections \( \nabla \) on \( M \) and sprays \( \sigma \) on \( M \).

5 **The Frobenius Theorem**

5.1 The Classical Frobenius Theorem and Proof

5.2 Differential Forms

infinitesimal cubes, exterior derivatives, Stokes theorem as an axiom—¿ proving exterior derivative is a graded derivation

5.3 The "amazing" right adjoint

5.4 Proof of the Frobenius Theorem

6 **Formulating the Cartan-Kähler Theorem Synthetically**

7 **Interpreting Synthetic Results Classically**

8 **Appendix**

Hadamard’s lemma, the \( \epsilon \)-neighborhood theorem

**References**


