Triangulations of the continuous cluster category $\mathcal{C}_\pi$

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joint work with Kiyoshi Igusa

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Triangulations of the continuous cluster category $C_\pi$
Goals of this talk

- Review the construction of the continuous cluster category $C_\pi$.
- Describe a classification of triangulations of $C_\pi$.
- Exhibit the three possible ”minimal” examples
Motivation

Why study the continuous cluster category $C_\pi$?

- Generalize cluster categories of type $A_n$
- General geometric interest: the disk model of the hyperbolic plane
Throughout this talk, let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $R$ be a discrete valuation ring with uniformizer $t$ and residue field $\overline{K} = K = R/(t)$, $\text{char}(K) \neq 2$.

**Definition**

A representation $V$ of $S^1$ over $R$ is given by an $R$-module $V[x]$ for each $[x] \in S^1$ and linear maps $V^{(x,\alpha)} : V[x] \to V[x - \alpha]$ for all $[x] \in S^1$ and $\alpha \in \mathbb{R}_{\geq 0}$ satisfying:

1. $V^{(x-\beta,\alpha)} \circ V^{(x,\beta)} = V^{(x,\alpha+\beta)}$
2. $V^{(x,2\pi n)} : V[x] \to V[x], m \mapsto t^n m$, $m \in V[x], \forall n \in \mathbb{N}$
Projectives representations of $S^1$

**Definition**

$P[x]$ is a representation of $S^1$ given by $P[x][x - \alpha] := Re_x^\alpha$ for $\alpha \geq 0$ and unique $R$-linear homomorphisms $P^{(x-\alpha,\beta)} : P[x][x - \alpha] \to P[x][x - \alpha - \beta]$ defined by $P^{(x-\alpha,\beta)}(e_x^\alpha) = e_x^{\alpha+\beta}$.

**Proposition**

$P[x]$ is projective and indecomposable for all $[x] \in S^1$. Any indecomposable is isomorphic to $P[x]$ for some $[x] \in S^1$. 

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Triangulations of the continuous cluster category $C_\pi$
The topology of $\text{Ind} \mathcal{P}_{S^1}$

**Definition**

By **topological $R$-category**, we mean a small category whose object and morphism sets are topological spaces and whose structure maps are continuous, including the $R$-module structure maps of the hom-sets.

Example: $\text{Ind} \mathcal{P}_{S^1}$ is a topological category: $\text{Ob}(\text{Ind} \mathcal{P}_{S^1}) \sim S^1$ and $\text{Mor}(\text{Ind} \mathcal{P}_{S^1}) = \{(r, x, y) | x \leq y \leq x + 2\pi\}/\sim$, where $\sim$ is defined by

- $(r, x, y) \sim (r, x + 2\pi, y + 2\pi), n \in \mathbb{Z}$
- $(r, x, x + 2\pi) \sim (tr, x, x)$

The morphism $(r, x, y)$ is defined by $e_x \mapsto re_y^{y-x}$.
Constructing $\mathcal{F}_\pi$ from $\mathcal{P}_{S^1}$

**Definition**

$\mathcal{F}_\pi$ is a category with objects given by pairs $(V, d)$, where $V \in \mathcal{P}_{S^1}$ and $d$ is an endomorphism with $d^2 = t$, and morphisms are $f : (V, d) \to (W, d')$ with $fd = d'f$.

**Theorem**

$\mathcal{F}_\pi$ is a Frobenius category.

Exact sequences of $\mathcal{F}_\pi$:

$$(X, d) \xrightarrow{f} (Y, d') \xrightarrow{g} (Z, d'') \iff 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is split exact in $\mathcal{P}_{S^1}$.  

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Triangulations of the continuous cluster category $C_\pi$
Proposition

1. $V \in \mathcal{P}_{S^1}$. Let $V^2 = \left( V \oplus V, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right)$. Then $V^2$ is projective-injective.

2. $\mathcal{F}_\pi$ Krull-Schmidt.

3. $\forall [x], [y] \in S^1$, $E(x, y) = \left( P_{[x]} \oplus P_{[y]}, \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right)$ is indecomposable.
**Proposition**

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2. \( \mathcal{F}_\pi \) is Krull-Schmidt.

3. \( \forall [x], [y] \in S^1 \), \( E(x, y) = \left( P_{[x]} \oplus P_{[y]}, \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right) \) is indecomposable.

Represent \([x]\) and \([y]\) by reals satisfying \( x \leq y \leq x + 2\pi \). Let \( \alpha = y - x \) and \( \beta = x + 2\pi - y \), giving morphisms \( \alpha_* : P_{[x]} \leftrightarrow P_{[y]} : \beta_* \) given by \( \alpha_*(e_x) = e_x^\alpha \) and \( \beta_*(e_y) = e_x^\beta \).
In the standard construction, $\text{Ob}(\text{Ind} \mathcal{F}_\pi)$ has the topology of a Möbius band, and the projective-injective objects residing on the boundary of the band.

This topology is preserved when we pass to the stable category $\mathcal{F}_\pi$, except that the boundary is excluded.

We may vary the construction of $\text{Ind} \mathcal{F}_\pi$ to be an even sheeted cover of the Möbius band.
$\text{Ind}\mathcal{F}_\pi$ usually has a single object from each isomorphism class. However...
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**Theorem (Igusa-Todorov)**

There is no way to continuously triangulate the stable category of $\text{add}(\text{Ind}\mathcal{F}_\pi)$ preserving the topology of the subcategory of indecomposables as an open Moebius band.
The topology of the stable category $\mathcal{F}_\pi$.

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Proof Sketch: one object in each isomorphism class of $\text{Ind}\mathcal{F}_\pi \Rightarrow X = \tau(X)$. Also, there is a continuous never zero path from morphisms $id_X$ and $f : X \rightarrow Y \Rightarrow \tau$ must be the identity functor.
We must pass to (at least) a 2-sheeted cover of the Moebius band. In fact...

**Theorem (G-Igusa)**

*Any cover of $\text{Ind}\mathcal{F}_\pi$ with an odd number of sheets does not admit a continuous triangulation.*
Constructing $C_\pi$

**Definition**

The **continuous cluster category** $C_\pi$ is the additive closure of the category with objects ordered pairs $X = (x_0, x_1) \in (S^1)^2$ with $x_0 < x_1 < x_0 + 2\pi$. $C_\pi(X, Y) = K$, when either

- $x_0 \leq y_0 < x_1 \leq y_1 < x_0 + 2\pi$ or
- $x_0 \leq y_1 < x_1 \leq y_0 + 2\pi < x_0 + 2\pi$ and 0 otherwise.

- $C_\pi$ isomorphic with stabilization of the additive closure of any $2m$-sheeted cover of $\text{Ind}F_\pi$.
- Clusters in $C_\pi$ are given by maximal discrete laminations of the hyperbolic plane (i.e. a family of non-crossing geodesics such that each has ”its own neighborhood”).
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- Let $C_n := \text{Ind}(\text{mod-}KQ_n)$, having $n$ isomorphic objects be a Schurian $K$-category with $C_n(j, i) = Kx_{ij}$. 
The categories $C_n$

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- Note $KQ_n$ is **not basic**, meaning it has simple modules of dimension greater than 1.
- By adding structure to $C_n$, we will construct triangulations of $C_\pi$.
Automorphisms of $\mathcal{C}_n$

- A set of bases $\{x_{ij}\}_{i,j \in [n]}$ is multiplicative if $x_{ij}x_{jk} = x_{ik}$.
- Any other set of bases $\{x'_{ij} = a_{ij}x_{ij}\}$, $a_{ij} \in K^*$, is multiplicative $\iff a_{ij}a_{jk} = a_{ik}$. This is a multiplicative system of scalars.

**Definition**

An automorphism $\sigma : \mathcal{C}_n \rightarrow \mathcal{C}_n$ is a $K$-linear functor that

- on objects $\sigma \in S_n$
- on morphisms is a multiplicative system of scalars $\{a_{ij}\}$ such that on basic morphisms $\sigma(x_{ij}) = a_{ij}x_{\sigma(i)\sigma(j)}$. 

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Triangulations of the continuous cluster category $\mathcal{C}_\pi$
Automorphisms of $\mathcal{C}_n$

**Theorem**

$\sigma \in \text{Aut}(\mathcal{C}_n)$. There is a set of multiplicative bases $\{x_{ij}\}$ so

- $a_{ij} = 1$ if $i$ and $j$ are in the same $\sigma$-orbit, and
- $a_{ik} = a_{jl}$ if $i$ and $j$ are in the same $\sigma$-orbit and $k$ and $l$ are in the same $\sigma$-orbit.

We call a set of bases such as in the above theorem **good bases**.
Fix \( \sigma \in Aut(C_n) \) and a set of good bases \( \{x_{ij}\} \) with respect to \( \sigma \).
An automorphism \( \tau \) of the pair \( (C_n, \sigma) \) is specified as follows:

- \( \tau \in S_n \) for objects of \( C_n \)
- \( \tau(x_{ij}) = b_{ij}x_{\tau(i)\tau(j)} \) with \( b_{ij} \) a multiplicative system
- \( \tau \sigma = \sigma \tau \). In terms of coefficients, \( b_{\sigma(i)\sigma(j)}a_{ij} = a_{\tau(i)\tau(j)}b_{ij} \).

\( \{b_{ij}\} \) are called the transition factors of \( \tau \). In general, it is not possible to find a set of bases which are good for both \( \sigma \) and \( \tau \).
In general,

- $Q$ a quiver with $p$ vertices; $Q_n$ has multiplicity $n_i$ at vertex $i$, where $n = (n_1, \ldots, n_p)$.
- Again, $KQ_n$ is not basic.
- Fixing good bases $\{x_{ij}\}$ for $(C_{n_r}, \sigma_r)$ and $\{y_{kl}\}$ for $(C_{n_s}, \sigma_s)$, we can write down conditions for the transition factors of $\alpha : (C_{n_r}, \sigma_r) \rightarrow (C_{n_s}, \sigma_s)$ ensuring continuity.
Given $\sigma$ an automorphism of $C_n$, let
\[ \mathcal{R} = \{(x, y) | x \leq y \leq x + 2\pi\} \subset \mathbb{R}^2. \]
Then
\[ \mu_\sigma = (\mathcal{R} \times C_n)/\sigma_* \]
where $\sigma_*(y, x + 2\pi, i) = (x, y, \sigma(i))$.

This is a covering category for the category of the open Moebius band.

$\sigma$ plays the role of a clutching map for the bundle that is an $n$-sheeted cover of the Moebius band over $S^1$. 

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Triangulations of the continuous cluster category $C_\pi$
Continuous automorphisms $\tau$

- $\sigma$ determines the topology of $\mu_\sigma$, so $\tau$ an automorphism of $(\mathcal{C}_n, \sigma)$ determines a continuous automorphism of the category $\mu_\sigma$, $\tau^*(x, y, i) = (y, x, \tau(i))$.
- $\mu_\sigma$ is algebraically equivalent to $\mathcal{C}_\pi$ and $\tau$ will play the role of a shift functor for $\mu_\sigma$.
- We need a way to specify distinguished triangles with respect to this triangulation functor to get a triangulation of $\mathcal{C}_\pi$. 

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Triangulations of the continuous cluster category $\mathcal{C}_\pi$
Example

\[ \sigma = (1234) \in Aut(C_4) \]
\[ \tau = \sigma^3 = (2341) \]

A schematic of \((\mu_\sigma, \tau)\)
Example

$\sigma = (1234) \in Aut(C_4)$

$\tau = \sigma^3 = (2341)$

$\phi : \sigma \rightarrow \tau$

A schematic of $(\mu_\sigma, \tau)$
The natural isomorphism $\phi$ for distinguished triangles

$\phi$ a $K$-linear, so $\phi_i = \cdot c_i$ for some $c_i \in K^*$. These constants are subject to some relations with the multiplicative system $\{a_{ij}\}$ of $\sigma$ and $\{x_{ij}\}$, and the transition factors $\{b_{ij}\}$ of $\tau$:

$$b_{ij}c_j = c_i a_{ij}, \quad c_{\sigma(i)} = -c_i a_{\sigma^{-1} \tau(i), i}$$

We define triangles in $\mu_\sigma$ as follows:

$$(x, y, i) \xrightarrow{1} (x, z, i) \xrightarrow{1} (y, z, i) \to (y, x, \tau(i))$$

where the last arrow is the composition

$$(y, z, i) \xrightarrow{1} (y, x + 2\pi, i) \sim (x, y, \sigma(i)) \xrightarrow{\phi} (y, x, \tau(i))$$
Theorem

There is a continuous triangulation of $\mathcal{C}_\pi$ for the data

- an even integer $n$
- pair of commuting automorphisms of $\mathcal{C}_n$, $\sigma$ and $\tau$
- natural transformation $\phi : \sigma \to \tau$ satisfying the conditions above.
- $\sigma(i)$ and $\tau(i)$ cannot reside in the same odd cycle of $\sigma$

All algebraically triangulated coverings of $\mathcal{C}_\pi$ arise in this way.
The case of 2 sheeted covers

not a triangulation

\( X = \sigma(X) \)

\( \sigma = id \)

\( \tau = id \)

\( \phi : a_{12} = -\frac{c_2}{c_1} \)

Orlov

\( \tau(X) \)

\( X = \sigma(X) \)

\( \sigma = id \)

\( \tau = (12) \)

\( \phi : b_{12} = -\frac{c_2}{c_1} \)

Igusa-Todorov

\( \sigma(X) = \tau(X) \)

\( \sigma = (12) \)

\( \tau = (12) \)

\( \phi : c_2 = -c_1 \)

a third triangulation

\( \sigma(X) \)

\( \sigma = (12) \)

\( \tau = id \)

\( \phi : c_2 = -c_1 \)

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Triangulations of the continuous cluster category \( C_\pi \)
The known triangulations

A very coarse classification: what’s isomorphic, or triangle equivalent?

Generalize to cluster categories of infinite rank not of type A, ”composition relations”

Key to understanding bundles of cluster categories of surface type?