Standing Waves: the Equilateral Triangle

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It is well known that symmetry considerations can often be a powerful tool for simplifying physical systems. We consider a wave which satisfies the Helmholtz equation in a region with some given symmetry, and note that any solution can be transformed by that symmetry to give another solution. Using this simple observation we describe a number of necessary conditions for wave solutions. For bounded regions, explicit solutions to the Helmholtz solutions are only known in a few cases (square, circle, and some special triangles). In this paper, we investigate the special domain of an equilateral triangle and describe what the solutions must look like based purely on symmetry considerations. Our analysis classifies all solutions based on their symmetry properties in a way that can be extended to any regular polygon. If one were to impose Dirichlet Boundary Conditions, then additional relationships within and between symmetry classes emerge. We then describe a collection of solutions which can be used to construct all of the solutions to the Dirichlet problem in the equilateral triangle.

I. INTRODUCING THE PROBLEM: CONFINED WAVES

There are a number of physical problems which involve the study of confined waves, including Quantum Billiards, Lasing modes for Nanostructures [3], and Vibrating Drum membranes. While the dispersion relations are different, the governing differential equations can all be separated and the spatial part can be rewritten as the Helmholtz Equation.

\[ \nabla^2 \psi + k^2 \psi = 0. \]  \hspace{1cm} (1)

It is important to note that this is a linear differential equation, (so if \(f\) and \(g\) are solutions, then so is any linear combination of \(f\) and \(g\)). Throughout this paper, we will refer to the constant \(k^2\) as the energy.

There are two different situations we are going to consider in this paper. At first we consider a more general case where the wave is confined to an region with the same symmetry as the equilateral triangle (120° rotational symmetry, and reflection symmetry), also known as the Dihedral group of order six, \(D_6\). In the second scenario, we take a closer look at the specific case of the equilateral triangle with Dirichlet boundary conditions (the solution vanishes along the boundary), and develop a number of relationships between different solutions.

In the more general case, we will make all our comments and constructions explicitly for the symmetry group \(D_6\), although the results generalize nicely to any region and its corresponding symmetry group. This shows the true power of using symmetry considerations, since analytic methods do not work for studying waves in regular polygons with more than 4 sides. In fact, the ground state solution in any polygon (other than those mentioned already) are known to be non-analytic at the vertices [5]. Additionally, some regions have been studied because when considered as a quantum billiard, the physical behavior is known to be chaotic[2]. A number of methods have been develop to approximate solutions in different 2 dimensional regions [4] [1].

In the special case of an equilateral triangle, we are able to use both symmetry and differential arguments to take a solution and construct solutions with different energies. In the end, we are able to generate all of the solutions of the equilateral triangle from its ground state solution (no nodal lines), and select solutions in the \((30°, 60°, 90°)\) triangle. This provides a partial converse to the common description of solutions in \((30°, 60°, 90°)\) triangles a consequence of solutions in the equilateral triangle with a nodal line along some altitude [1].

Our analysis starts in Section II by introducing a few notions from Representation Theory and the study of Inner Product Spaces. We will describe the irreducible representations of the group \(D_6\), and the \(L^2\) norm on the space of solutions. This will also be the setting to introduce the guiding role the symmetry of the boundary plays to restrict the solutions. In Section III, the specific classes of symmetries are defined, and their connection with representation theory revealed. Up to this point, no assumptions are made about specific types of boundaries. However, starting in Section IV we develop the interconnections between the symmetry classes in the case of the equilateral triangle. Finally, in Section V we bring together our results into a coherent picture of the equilateral triangle and concludes with some additional topics for future study.

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II. MATHEMATICAL BACKGROUND

There are two important subjects from Mathematics which are vital to the description given below. The first is called Representation Theory, which gives a way of importing all of the information from our symmetry group. The second is the notion of an inner product space, which generalizes vector spaces and will be a useful tool for keeping track of our solutions.

A. Representation Theory

The idea of a representation is to capture the properties of an abstract group in a convenient and concrete way. Formally, it is a homomorphism \( \rho \) from the group \( G \) into the group of linear transformations of a vector space (in our case, the real numbers suffice); \( \rho : G \rightarrow \text{GL}_n(\mathbb{R}) \).

This assigns to each group element a transformation of the vector space that is consistent with the multiplication table of the group. For \( D_6 \), every such representation can be decomposed into a direct product of three so-called irreducible representations. These irreducible representations turn out to determine the symmetry of our solutions exactly.

Before we can describe these homomorphisms, we need to describe the elements of the group \( D_6 \). Let \( \sigma \) be a \( 120^\circ \) counter-clockwise rotation, and \( \mu \) be a reflection. The defining relationship of the dihedral group says that \( \mu \sigma = \sigma^{-1} \mu \). Since these two elements generate the whole group, we only need to define each homomorphism on these generators. Listing the elements of the group, \( D_6 = \{ e, \sigma, \sigma^2, \mu, \mu \sigma, \mu \sigma^2 \} \).

The first irreducible representation is called the trivial representation because it maps every group element to the identity map of \( \mathbb{R} \). While this may seems somewhat useless, it actually plays an interesting role later on. Symbolically, \( \rho_1(g) = 1 \) for every \( g \in D_6 \).

The second representation can be thought of as the orientation representation, because it distinguishes between whether or not a reflection occurred. That is, \( \rho_2(\sigma) = 1 \), but \( \rho_2(\mu) = -1 \). This representation is also called the sign representation. Note that these first two representations are one dimensional (because they take values in \( \text{GL}_1(\mathbb{R}) \cong \mathbb{R} \)).

The final representation is the only representation that distinguishes every nuance in the group, and is therefore sometimes used as a definition of \( D_6 \). Unlike the previous representations, \( \rho_3 \) is called a two dimensional representation because it takes values in \( \text{GL}_2(\mathbb{R}) \), or \( 2 \times 2 \) invertible matrices, of which the following two defining matrices should look familiar:

\[
\rho_3(\sigma) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \rho_3(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Using this last representation of the group, we can define in a natural way the action of a group element \( g \) on a solution to the Helmholtz Equation \( f(x, y) \). The argument of the function is interpreted as the vector \( \begin{pmatrix} x \\ y \end{pmatrix} \) in \( \mathbb{R}^2 \), so the following action is well defined.

\[
(g \cdot f)(x, y) = f(\rho_3(g)^{-1} \begin{pmatrix} x \\ y \end{pmatrix})
\]

The use of the inverse of the representation matrix is required to make this a homomorphism, and can be thought of as a passive transformation on the coordinates. We will implicitly use this third representation throughout this paper to see what happens to a solution when it is transformed by a group element \( g \), although we will often write \( gf \) in place of \( g \cdot f \).

B. Inner Product Spaces

In Linear Algebra, nearly everything developed rested on the notion of an inner product, including the notions of length and orthogonality. We are most interested in developing these notions for solutions to the wave equation. To do this, consider two solutions \( f_1 \) and \( f_2 \). Then we define

\[
\langle f_1, f_2 \rangle = \int \int_{\Delta} f_1(x, y) f_2(x, y) dx dy
\]

Where the integral is taken over the domain \( \Delta \). Using this inner product, we can construct a norm for a solution. Suppose \( f \) is a solution, then its norm (or length) is

\[
||f|| = \sqrt{\langle f, f \rangle}.
\]

This is sometimes called the \( L^2 \) norm, and we will use it to normalize a solution.

We can also use the inner product to determine when two solutions are orthogonal (or perpendicular). If \( \langle f_1, f_2 \rangle = 0 \), then we say that \( f_1 \) and \( f_2 \) are orthogonal solutions (denoted \( f_1 \perp f_2 \)). If two solutions have the same energy and are orthogonal, we can consider a two dimensional ‘solution space’ spanned by these solutions.

In the same way that we use the vector \( \begin{pmatrix} a \\ b \end{pmatrix} \) to represent \( a \hat{i} + b \hat{j} \), in our context it will represent \( af_1 + bf_2 \). Thus the language of Linear Algebra can be used to describe this solution space.

We will often be able to quickly conclude that two solutions are orthogonal by using a trick that is often introduced in introductory calculus classes. Since we have reflection symmetry about the \( x \)-axis, the integration domain is symmetric in \( y \). Thus if the function \( f_1 f_2 \) is odd in \( y \), then we can conclude immediately that \( f_1 \perp f_2 \).

III. SYMMETRY CLASSES

We now have the notation necessary to describe how symmetry will influence the solutions. Suppose \( f \) is a
solution to the Helmholtz Equation (1). Then for any
element \( g \in D_3 \), \( gf \) is a solution with the same value of \( k \)
(i.e. same energy), where we use the action of the group
defined in Eqn. [2]. In the following subsections, we are
going to organize all of the solutions into sets where each
element of the set has the same symmetry. We call these
sets \textit{symmetry classes}.

We will first consider the affect of rotating the solution,
and compare the original solution \( f \) to the new solution
(which is denoted \( \sigma f \) in our notation). One of the fol-
lowing things can happen: Either \( \sigma f = f \) (rotationally
symmetric), \( \sigma f = -f \) (rotationally anti-symmetric), or
\( \sigma f \neq \pm f \) (rotationally asymmetric). We can eliminate
the rotationally anti-symmetric case as follows. Suppose
that \( \sigma f = -f \), then using the fact that \( \sigma^3 \) is the identity
element of the group,

\[
  f = \sigma^3 f = \sigma (\sigma (-f)) = \sigma f = -f.
\]

Thus \( f(x) = -f(x) \) for every \( x \in \Delta \), so \( f \) is identically
zero on the domain, and therefore uninteresting.

1. Rotationally Symmetric Solutions

We continue by considering the rotationally symmetric
solutions. In addition to considering rotations, we also
need to consider the affect of the other generator of our
symmetry group, the reflection \( \mu \). As before, any given
solution is either symmetric, anti-symmetric, or asymmetric.
However, we do not need to consider the asymmetric
case. Suppose that we have a solution \( f \) so that
\( \mu f \neq \pm f \), we can create the following two functions:

\[
  f_+ = \frac{1}{2}(f + \mu f) \quad \text{and} \quad f_- = \frac{1}{2}(f - \mu f).
\]

These are also solutions by linearity of Eqn. 1. Notice
that using these new functions we can write the original
function \( f = f_+ + f_- \). Furthermore, these new func-
tions are either symmetric \( f_+ = \mu f_+ \) or anti-symmetric
\( f_- = -\mu f_- \), and are not trivial by assumption. Fur-
thermore, any uniform boundary condition on \( f \) will be
passed along to \( f_+ \) and \( f_- \). So we were able to decom-
pose an asymmetric solution into the sum of a symmetric
and anti-symmetric solutions. Since we can do this
in general, we need only consider symmetric and anti-
symmetric solutions under reflection. Denote these two
symmetry classes by \( \text{A1 and A2} \), respectively. Notice
that for any solution \( f_1 \in \text{A1} \), \( \mu f_1 = f_1 \) means that
\( f(x, -y) = f(x, y) \), and so we say that \( f_1 \) is even in
\( y \). Alternatively if \( f_2 \in \text{A2} \) then \( \mu f_2 = -f_2 \) means that
\( f(x, -y) = -f(x, y) \), and so \( f_2 \) is odd in \( y \). As a result,
the product \( f_1 f_2 \) is odd in \( y \), and so \( f_1 \perp f_2 \) by the
discussion at the end of Subsection II B.

\[
\begin{array}{c|c|c}
  f_1 \in \text{A1} & \sigma f_1 & \mu f_1 \\
  f_2 \in \text{A2} & +f_1 & +f_2 \\
  & +f_2 & -f_2 \\
\end{array}
\]

This can be made much more concise using the irre-
ducible representations from Section (II A). Notice that
\( f_1 \in \text{A1} \) is characterized by the trivial representation:

\[
  gf_1 = \rho_1(g) f_1,
\]

and \( f_2 \in \text{A2} \) is characterized by the sign representation

\[
  gf_2 = \rho_2(g) f_2.
\]

To help visualize these symmetry classes, consider Fig.
1 for examples of solutions with the desired symmetry.

2. Rotationally Asymmetric Classes

If a solution is not rotationally symmetric, then all
we can say about it is whether it is symmetric or anti-
symmetric with respect to \( \mu \). Name these classes E1 and
E2, respectively. While at first the lack of symmetry
may make these solutions seem out of place, these are
actually the most common solutions and have a number of
interesting properties. Consider a normalized solution
\( f_1 \) in class E1 (that is, \( \mu f_1 = f_1 \)). Then at this point
we do not know anything about \( \sigma f_1 \) except that it is a
solution with the same energy. For that matter, so is
\( \sigma^2 f_1 \). So let’s consider the function \( \hat{f}_2 = \sigma f_1 - \sigma^2 f_1 \). We
claim that \( \hat{f}_2 \) is in symmetry class E2.

\[
  \mu \hat{f}_2 = \mu (\sigma f_1 - \sigma^2 f_1) = \mu \sigma f_1 - \mu \sigma^2 f_1 = \sigma^2 f_1 - \sigma f_1 = -\hat{f}_2.
\]

The reason we have denoted this new solution \( \hat{f}_2 \) is be-
because at this point, we have no reason to believe that \( \hat{f}_2 \)
is a better choice than say, \( 2 \cdot f_2 \). To settle this point,
we use the discussion from Section II B to normalize \( \hat{f}_2 \),
which we will call \( f_2 \). We also take this opportunity to
point out that $f_1$ and $f_2$ must be orthogonal, since their product is odd in $y$.

Since the action of the group introduces a second solution, it is reasonable to consider the two dimensional solution space spanned by $f_1$ and $f_2$. As in Section II B, the vector $\vec{f} = \begin{bmatrix} a \\ b \end{bmatrix}$ represents the solution $a f_1 + b f_2$. Written in this way, we can recognize the third irreducible representation $\rho_3$.

$$g \vec{f} = \rho_3(g) \vec{f}.$$  

This equation actually contains a lot of information, and is strikingly similar to Equations 3 and 4. We will use it to find the proper normalization constant that relates $f_1$ to $f_2$.

Start with $\vec{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, representing $f_1$. Then $\sigma f$ can be re-expressed as a linear combination of $f_1$ and $f_2$ as follows:

$$\sigma f = \sigma f_1 = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -\sqrt{3} \end{bmatrix} = \frac{1}{2} f_1 + \frac{\sqrt{3}}{2} f_2$$

Doing this again for $\sigma^2 f$, we get

$$\sigma^2 f = \sigma^2 f_1 = -\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2$$

So $\sigma f_1 - \sigma^2 f_1 = \sqrt{3} f_2$. Solving for $f_2$, we get

$$f_2 = \frac{1}{\sqrt{3}} (\sigma f_1 - \sigma^2 f_1).$$

It is not surprising that we can also express $f_1$ in terms of $f_2$ using an identical method.

$$f_1 = \frac{1}{\sqrt{3}} (\sigma f_2 - \sigma^2 f_2).$$

Note that for both cases the solution on the left hand side must not be trivial. If it were, then we would get that $f_i$ was rotationally symmetric, contradicting our assumptions.

To help visualize symmetry classes E1 and E2, consider Fig. 2 for examples of solutions with the desired symmetry.

To recap, here are the four symmetry classes for solutions.

<table>
<thead>
<tr>
<th>Reflection</th>
<th>Symmetric</th>
<th>Asymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric</td>
<td>A1</td>
<td>E1</td>
</tr>
<tr>
<td>Anti-Symmetric</td>
<td>A2</td>
<td>E2</td>
</tr>
</tbody>
</table>

We have already seen that the solutions in symmetry classes E1 and E2 are intimately related, and in general this is all we can say about arbitrary regions with $D_6$ symmetry. In the next section we consider the special case where the boundary is precisely an equilateral triangle.

IV. GENERATING NEW SOLUTIONS FROM OLD

In this section we will focus on how to take a known solution and create an orthogonal solution with an explicit relationship between the two energies. First we will build up a correspondence between A2 and E2, and then find a way to transform symmetry type A2 to type A1. Finally, there are ways of taking any solution describing a higher harmonic, creating higher energy waves which are usually in the same symmetry class (Rotationally asymmetric solutions may become rotationally symmetric).

A. E2 $\leftrightarrow$ A2

A recurring theme in the next few sections will be to consider the solution within the triangle as it would extend outside. To do this, we want to take the triangle that we have been working with, and refer to it as a fundamental domain. Reflecting the fundamental domain across each boundary, we can tessellate the plane. This is shown in Fig. 3, along with the image of a point in the fundamental domain under each reflection. It is not hard to see that this smoothly extends the solution to the whole plane.

Now consider a solution with E2 symmetry. Note that it must be zero along the $x$ axis and that the top half must be the reflected image of the bottom half. Thus without loss of generality, we can picture a solution of type E2 will actually look like the one in Fig. 2(b), with possibly more nodal curves.
the above analysis still works to give us that solution of class A2, we can transform it to a new solution from the symmetry class A2. On our earlier discussion, if although the energy will triple. That is, can confirm that if is given below:

\[ f \circ T^{-1} \] which will have symmetry either A2 or E2. By repeating this process the energy of the solution will continue to decrease. Since the ground state has the lowest energy, this process must stop with a solution with E2 symmetry. This gives us a clear correspondence between solutions with A2 and E2 symmetry.

**B. A2 to A1**

Suppose \( f_2 \) is a solution with A2 symmetry, and we want to find a solution with the same energy but in symmetry class A1. Start by realizing that one big difference is that \( f_2(x, y) \) is odd in \( y \) and we need to make it even in \( y \). To do this, define the following intermediate function.

\[ \tilde{f}_1(x, y) = \frac{d^3}{dy^3} f_2(x, y) \]

Note that since we are dealing with a differential equation, \( \tilde{f}_1 \) still satisfies the Helmholtz equation (with the same energy) using Clairaut’s Theorem that partials commute.

While this makes the function even in \( y \), the rotational symmetry is now gone. To resolve this, we create a new function \( f_2 \) which is forced to be rotationally symmetric.

\[ \tilde{f}_1(x, y) = \tilde{f}_1(x, y) + \sigma \tilde{f}_1(x, y) + \sigma^2 \tilde{f}_1(x, y) \]

This new function is, by design, of symmetry class A1. With some extra work, it can also be shown that the boundary conditions are preserved and that it is not trivial. Thus we can normalize it to get the desired solution \( f_1 \) with the same energy as \( f_2 \). While this can be confirmed explicitly using the known solutions, see Future Work (Section V) for more.

Note that taking the first derivative would also create an intermediate function that is even in \( y \), but the solutions would vanish once it is symmetrized.

This process can be reversed, however for an arbitrary solution from symmetry class A1 we cannot guarantee that this transformation gives a non-trivial solution. This is not entirely a surprise, since not all solutions with symmetry class A1 live in two dimensional solutions spaces (for example, the ground state with no nodal lines). In order to fully understand the \( A1 \leftrightarrow A2 \) correspondence, we need to know all of the solutions with A1 symmetry which become zero. To do this, we will need the language of the next section.

**C. Higher Harmonics**

Similar to the method described in Section IV A, there is actually a more direct way to take a solution and create a solution with higher energy. This is by recognizing that for any \( n \in \mathbb{N} \) an equilateral triangle can be decomposed into \( n^2 \) equilateral triangles. So, by a simple dilation and a translation along the \( x \)-axis any solution
FIG. 5: The equilateral triangle can be decomposed into \( n^2 \) equilateral triangles

(a) \( n = 2 \)  
(b) \( n = 3 \)

FIG. 6: These pictures show two redundant constructions of a higher energy state.

(a) Two successive constructions from Section IV A  
(b) Harmonic with \( n = 3 \)

can be used to construct a higher harmonic (with higher energy), with \( k^2 \to n^2 k^2 \). If the solution started with rotational symmetry, then this will not change the symmetry class. However, if it started asymmetric it may become rotationally symmetric. This will only happen if \( n \) is divisible by 3, in which case we have would have already constructed this solution by using the method from Section IV A) twice. See Fig. 6 to see the equivalent constructions.

V. CONCLUSION AND FUTURE WORK

Now for a way to bring this all together. Going back to the A1 \( \leftrightarrow \) A2 correspondence, it is possible to show that the only solutions of type A1 which become zero under the transformation described in Section IV B are precisely the ground state and its harmonics (as described in Sections IV C).

Putting this all together, we can generate all the solutions in the equilateral triangle (except the ground state and its harmonics) by starting with a solution in the \( 30^\circ - 60^\circ - 90^\circ \) triangle, extending it to a solution of symmetry type E2, and then either transform it into type E1 (Section III 2) or A2 (Section IV A). Repeatedly transforming these solutions by the methods from Section IV A and Section IV C gives us all of symmetry class A2. Finally, these solutions from A2 can be transformed to A1 (Section IV B).

To see that this does indeed describe all solutions, the following algorithm gives a way to take any solution and describe the generating solution (either from the \( 30^\circ, 60^\circ, 90^\circ \)) through a sequence of reversible steps.

1. Given an arbitrary solution to the Dirichlet Problem in the equilateral triangle, determine its symmetry class (one may need the process described in Section III 1 to get reflection (anti)symmetry).

2. Reduce the energy as much as possible using the method from Section IV C. This can be done if there is any vertical nodal line.

3. If the solution is of type E2, then extract the solution to the \( 30^\circ - 60^\circ - 90^\circ \) triangle.

4. If the solution is of type E1, then perform the operation from Section III 2 to get a solution of type E2, go to step 3.

5. If the solution is of type A2, then perform the operation from Section IV A repeatedly until you get a solution of type E2, then go to step 3.

6. If the solution is of type A1, then perform the operation from Section IV B.

   (a) If you obtain a solution of type A2, go to step 5.

   (b) If you obtain the zero solution, then you previously had the ground state solution.

There are several areas for future work. First, we intend to extend some of the results in this work to a regular \( n \)-gon and the limiting case as \( n \to \infty \). To do this we will also present the A1 \( \leftrightarrow \) A2 in a more robust theoretical framework so that this method can stand on its own. If one considers more generally the action of the ring of differential operators on the vector space of solutions, the process we have used is none other than the sign quasi-invariant in the polynomial ring \( \mathbb{Z}[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] \).

Secondly, one can investigate the analogous procedure for Neumann boundary conditions and determine if the result there are more illuminating for more general boundary conditions.

Finally, these results may also be implemented in the numerical determination of the solutions as a way to increase the rate of convergence. This will be more useful in domains where an explicit solution is not known.

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