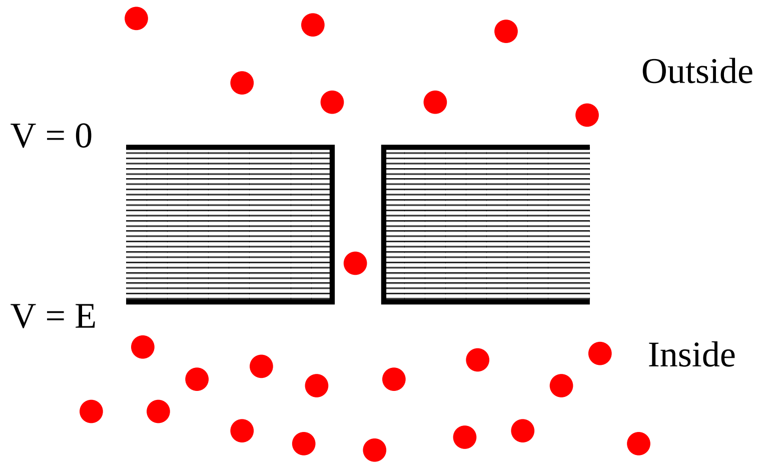
Leaky Integrate-and-Fire Model Neuron

Nernst Equation and Equilibrium/Reversal Potential

Considering one type of ion (diagram is typical of Potassium, K) with negative reversal potential, $E_K \approx -90mV$.



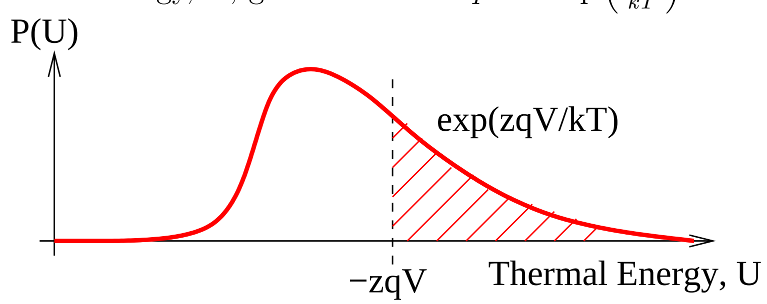
Pumps (requiring ATP and without transferring charge) ensure concentration of potassium $[K^+]$ is greater inside than outside the cell.

Potassium ions (carrying charge) would on average flow out of the cell through ion channels, leaving the inside at a negative potential ($E_K < 0$) until a dynamic equilibrium is reached where the flow in matches the flow out.

Rate of flow in either direction is proportional to concentration at starting point.

But only a fraction of ions with enough energy can make it out once there is a potential difference across the membrane.

Fraction with thermal energy, U, greater than $-zqV$ is $\exp\left(\frac{zqV}{kT}\right)$.



For one ion equilibrium potential E is where $[inside]\exp\left(\frac{zqE}{kT}\right) = [outside]$. Hence

$$E = \frac{kT}{zq}\ln\left(\frac{[outside]}{[inside]}\right) \qquad \textit{Nernst Equation} \tag{1}$$

Each ion has its own E. Calculation of equilibrium for all types of channel leads to a reversal potential or equilibrium potential or leak potential, V_L, of the cell, where overall charge flow is zero. $V_L \approx -70mV$.

Membrane potential, V_m

If $V_m > V_L$ (that is we **depolarize**, make less negative) charge flows out of the cell and the membrane potential potential goes back down (becomes more negative).

If $V_m < V_L$ (that is we **hyperpolarize**, make more negative) charge flows into the cell and the membrane potential potential goes back up (becomes less negative).

Current flow per unit area **out** of the neuron through "leak" channels in the membrane is given by $I_m = G_L\,(V_m - V_L)$. (Positive membrane current is outward.)

Change in potential due to current flow depends on the membrane capacitance, C_m, via $Q = C_m V_m$ where Q is excess charge inside the neuron.

Since $dQ/dt = -I_m$ we have in general:

$$C_m \frac{dV_m}{dt} = -G_L\,(V_m - V_L) + I_{app} \qquad (2)$$

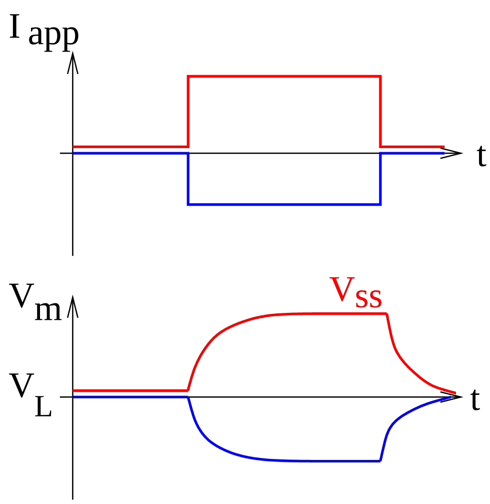
where I_{app} is an externally applied (inward) current. (Positive applied current is inward.)

c_m is specific membrane capacitance = capacitance per unit area. Total capacitance, $C_m = c_m A$.

r_m is specific membrane (surface) resistance, Total input resistance, $R_{in} = r_m/A$.

$g_L = 1/r_m$ is specific membrane conductance. Total conductance, $G_L = g_L A$.

Response to a long current pulse



Time course is exponential decay to a steady state (V_{ss}) with a time constant τ_m.

Proof: rewrite Eq. 2 as

$$\frac{dV_m}{dt} = \frac{V_{ss} - V_m}{\tau_m} \quad \text{where} \qquad (3)$$

$$\tau_m = \frac{C}{G_L} = CR_{in} \approx 20ms \qquad (4)$$

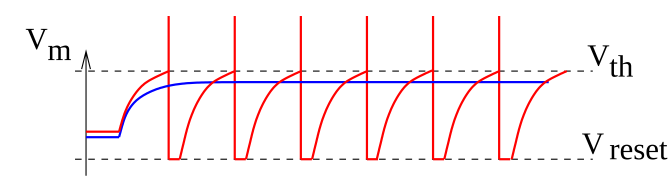
$$\text{and} \quad V_{ss} = V_L + I/G_L \qquad (5)$$

If current switches at time, $t = 0$ then solution is:

$$V_m(t) = V_{ss} + [V_m(0) - V_{ss}]\,\mathrm{e}^{-t/t_m}. \qquad (6)$$

Leaky integrate-and-fire

We have already shown leaky integration. Now add a 'fire' by hand when V_m reaches a threshold, V_{th} then reset to V_{reset} and wait a short refractory time, τ_{ref} before further integration.



What is the firing rate, $f(I)$? Time between spikes is $\tau_{ref} + T$ where T is time for V_m to increase from V_{reset} to V_{th}.

Again, integrating Eq. 3:

$$\int_{V_{reset}}^{V_{th}} \frac{dV_m}{V_{ss} - V_m} = \int_0^T \frac{dt}{\tau_m} \qquad (7)$$

leads to

$$T = -\tau_m \ln\left[\frac{V_{ss} - V_{th}}{V_{ss} - V_{reset}}\right] \qquad (8)$$

Then $f(I) = 1/\left(\tau_{ref} + T\right)$ **if** $I > I_c$.

I_c defined by $V_{ss} = V_{th}$ leads to $I_c = G_L(V_{th} - V_L)$.

Euler method for ODEs

Cf *Numerical Recipes* by W.H.Press *et al.*
See also: *Modeling in Biology: Differential Equations* by Clifford Taubes, Prentice Hall, 2001.
To solve numerically:

$$\frac{dx}{dt} = f(x) \qquad \text{with} \qquad x_0 = x(t=0) \tag{9}$$

Write $x_n = x(t = n\Delta t)$. Define: $\Delta x_{n+1} = x_{n+1} - x_n$ then for small Δt Eq. 9 becomes

$$\frac{\Delta x_{n+1}}{\Delta t} = f(x_n) \tag{10}$$

hence

$$x_{n+1} = x_n + f(x_n)\Delta t \qquad \text{error per step} \qquad \approx (\Delta t)^2 \tag{11}$$

Euler method with white noise

White noise, $w(t)$ is defined as having zero mean, but a variance that is a delta-function in time.

$$\langle w(t) \rangle = 0 \qquad ; \qquad \langle w(t)w(t') \rangle = \delta(t - t') \tag{12}$$

The integral (dt) over a delta-function gives, unity, so the delta-function has units of inverse time (ms^{-1}) and the white noise function has units of square-root of inverse-time, ($\mathrm{ms}^{-1/2}$).

Equation to integrate:

$$\frac{dx}{dt} = f(x) + \sigma w(t) \tag{13}$$

Must be implemented numerically:

$$x_{n+1} = x_n + f(x_n)\Delta t + \sigma\sqrt{\Delta t}\tilde{w}_n \tag{14}$$

where $\tilde{w}_n$ is a random selection from a Gaussian, with distribution:

$$p\left(\tilde{w}_n\right) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\tilde{w}_n^2/2} \tag{15}$$

This has zero mean, and variance 1 for a particular n but covariance 0 for terms with different n: this can be written $\langle \tilde{w}_n \tilde{w}_{n'} \rangle = \delta_{n-n'}$.

Warning: only read further if everything up until now is straightforward to you!

Why scale noise term by $\sqrt{\Delta t}$ in simulations?

In words: the white noise term adds variance to x, not a mean effect.

For N independent choices of noise term (one for each time step) if the amplitude of the noise term is A then the variance of N terms is NA^2. Hence at time $T = N\Delta t$ the variance is NA^2.

If we change Δt the number of time steps, N to reach T changes inversely, $N = T/(\Delta t)$. But we do not want the variance to change as we change the time step of our simulation, so we want NA^2 to remain fixed. This means A^2 scales as $1/N \propto \Delta t$. Recall A is the amplitude of the noise term, so the amplitude scales as $A \propto \sqrt{\Delta t}$.

Mathematical proof: just consider contribution of the noise term, which only depends on time, so we ignore $f(x)$ in Eq. 13. If:

$$\frac{dx}{dt} = \sigma w(t) \tag{16}$$

then

$$x(T) = x_0 + \sigma \int_0^T w(t')dt' \tag{17}$$

and $\langle x(t) \rangle = x_0$ while

$$\begin{aligned} Var(x) = \left\langle (x - x_0)^2 \right\rangle &= \left\langle \sigma^2 \int_0^T w(t')dt' \int_0^T w(t'')dt'' \right\rangle \\ &= \sigma^2 \int_0^T dt' \int_0^T dt'' \langle w(t')w(t'') \rangle \\ &= \sigma^2 \int_0^T dt' \int_0^T dt'' \delta(t' - t'') \\ &= \sigma^2 \int_0^T dt' = \sigma^2 T. \end{aligned} \tag{18}$$

This is the analytic solution (using continuous time) that we must achieve numerically (using discrete time steps).

Now we prove that Eq. 14, ignoring the $f(x_n)$ term, gives the correct result:

$$x_{n+1} = x_n + \sigma\sqrt{\Delta t}\tilde{w}_n \tag{19}$$

leads to

$$x_N = x_0 + \sigma\sqrt{\Delta t} \sum_{k=1}^{N} \tilde{w}_k \tag{20}$$

where the sum is over N independent variables, each with mean 0 and variance 1 so $\langle x_N \rangle = x_0$ and

$$\begin{aligned} Var(x_N) = \left\langle (x_N - x_0)^2 \right\rangle &= \sigma^2 \Delta t \sum_{k=1}^{N} \sum_{k'=1}^{N} \langle \tilde{w}_k \tilde{w}_{k'} \rangle \\ &= \sigma^2 \Delta t \sum_{k=1}^{N} \sum_{k'=1}^{N} \delta_{k-k'} \\ &= \sigma^2 \Delta t \sum_{k=1}^{N} 1 \\ &= \sigma^2 \Delta t N = \sigma^2 T \qquad \text{where} \qquad T = N\Delta t. \end{aligned} \tag{21}$$

This agrees with Eq. 18 as required.