Ordinary Differential Equations and the Exponential Function

**Computer solution for a general equation by Euler method:**

The general equation: \( \frac{dN}{dt} = f(N,t) \)

where \( f(N,t) \) can be any function under the sun, is solved by starting with some initial value, \( N(1) = N(t=0) \) and updating with:

\[
N(i + 1) = N(i) + (dt)f[N(i),t(i)] \quad \text{where} \quad t(i + 1) = t(i) + (dt)
\]

(Equation 1).

**Notes:** (see Matlab code xvode.m).

Choose a small value for the time step “\( dt \)”: always check that “\( dt \)” is small enough by comparing with the exact solution or comparing with a code using a much smaller value for “\( dt \)”.

Set the initial value.

Set a time vector, \( t \), running from 0 to \( t_{\max} \) in steps of \( dt \)

Start a “for” loop with an index (eg “\( i \)” running from 2 to \( \text{length}(t) \).

At each time step, update the variable \( N \) using:

“change in N” is equal to “change in time” multiplied by \( f(N,t) \)

which is the meaning of Equation 1.

**An alternative method for more stable computational solution:**

Solve the equation analytically assuming the variables are fixed, before stepping forward in time.

For example, a linear ODE can be integrated as:

\[
N(i + 1) = N_{SS} + \left[ N(i) - N_{SS} \right]\exp\left[\frac{-dt}{\tau}\right]
\]

where \( N_{SS} \) and \( \tau \) are defined above. We use this method to stabilize some codes.

For those of you who work on the BZ problem, the analytic solution for a quadratic ODE has been used to update the concentration of HBrO₂ \( (X) \) to allow for robust simulations. Otherwise, the time step \( dt \) has to be so small that simulations lasting several oscillating cycles take way too long.
Background Math: General Mathematical Proof of Exponential Solution for Linear ODE.

If you can write:
\[
\frac{dN}{dt} = \frac{N_{ss} - N}{\tau}
\]
then:
\[
\frac{dN}{N_{ss} - N} = \frac{dt}{\tau}
\]
integrate:
\[
\int_{N(0)}^{N(t)} \frac{dN}{N_{ss} - N} = \frac{1}{\tau} \int_0^t dt
\]
to give:
\[
-\ln \left[ \frac{N_{ss} - N(t)}{N_{ss} - N(0)} \right] = \frac{t}{\tau}
\]
which means:
\[
\frac{N_{ss} - N(t)}{N_{ss} - N(0)} = \exp \left( -\frac{t}{\tau} \right) \text{ and } N(t) = N_{ss} - [N_{ss} - N(0)] \exp \left( -\frac{t}{\tau} \right)
\]