DIRICHLET SERIES

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We give here an outline of some of the basic properties of Dirichlet L-series. We commence by defining a Dirichlet L-series for a Dirichlet character χ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where $s = \sigma + it$ and $L(s,\chi)$ is defined as above for $\sigma > 1$. We note that $L(s,\chi)$ converges absolutely for $\sigma > 1$. Also, $L(s,\chi)$ converges uniformly in every half plane $\sigma \geq 1 + \delta$ where $\delta > 0$, so we have that $L(s,\chi)$ is an analytic function of s in the half plane $\sigma > 1$.

Euler Product Expansion. We have the following Euler product expansion of the Dirichlet L-series $L(s,\chi)$:

$$L(s,\chi) = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

over all primes p. This Euler product is also convergent for $Re(s) = \sigma > 1$.

We note that if $\chi = \chi_1$ (principal character mod k), then since $\chi_1(p) = 0$ if $p|k$ and $\chi_1(p) = 1$ if $p \nmid k$,

$$L(s,\chi_1) = \prod_{p \nmid k} \frac{1}{1 - \frac{1}{p^s}} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \prod_{p|k} 1 - \frac{1}{p^s} = \zeta(s) \prod_{p|k} 1 - \frac{1}{p^s}.$$

So $L(s,\chi_1)$ for the principal character χ_1 is just the Riemann zeta function $\zeta(s)$ multiplied by a finite number of factors.

Hurwitz Zeta Functions. We generalize the Riemann zeta function to the Hurwitz zeta functions $\zeta(s,a)$ defined for $\sigma > 1$ by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

where $a \in \mathbb{R}$ is fixed and $0 < a \leq 1$. We note that the Riemann zeta function is the particular case of the Hurwitz zeta function where $a = 1$. Just like the Dirichlet L-functions, the Hurwitz zeta functions $\zeta(s,a)$ converge absolutely for $\sigma > 1$. Also analagous to the Dirichlet L-functions, $\zeta(s,a)$ converges uniformly in every half plane with $\sigma \geq 1+\delta$, $\delta > 0$ and hence $\zeta(s,a)$ is analytic for $\sigma > 1$.

We can write Dirichlet L-functions in terms of Hurwitz zeta functions as follows. If χ is a character mod k, then let $n = qk + r$, where $1 \leq r \leq k$ and $q = 0, 1, 2, \ldots$. Then

$$\begin{aligned} L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} &= \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{\chi(qk+r)}{(qk+r)^s} \\ &= \frac{1}{k^s} \sum_{r=1}^{k} \chi(r) \sum_{q=0}^{\infty} \frac{1}{(q+\frac{r}{k})^s} \\ &= k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}). \end{aligned}$$

Analytic Continuation. Using the gamma function $\Gamma(s)$, we can get an integral representation for the Hurwitz zeta functions. This integral representation can then be used to obtain a contour integral representation of the Hurwitz zeta function. We then write the Hurwitz zeta function as a product of the gamma function and a contour integral. Since both the gamma function and the contour integral are valid for all $s \in \mathbb{C}$, we use this as the definition of $\zeta(s,a)$. Using this definition of $\zeta(s,a)$, it is possible to show that $\zeta(s,a)$ is analytic for all s except at $s = 1$, where it has a simple pole with residue 1.

We now define $L(s,\chi)$ by

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}).$$

This gives us the analytic continuation of $L(s,\chi)$ beyond the line $\sigma = 1$. We find that $L(s,\chi_1)$ is analytic everywhere except for a simple pole at $s = 1$ with residue $\phi(k)/k$ for the principal character χ_1 mod k and if χ is not the principal character χ_1, then $L(s,\chi)$ is entire.

Functional Equation. Before giving the functional equation for Dirichlet L-series, we note that it suffices to give a functional equation for *primitive characters mod* k. For any Dirichlet character χ mod k with d any induced modulus, we can write $\chi(n) = \psi(n)\chi_1(n)$ where ψ is a character mod d and χ_1 is the principal character mod k. Then for all s,

$$L(s,\chi) = L(s,\psi) \prod_{p|k} \left(1 - \frac{\psi(p)}{p^s}\right).$$

We can take d to be the conductor of the Dirichlet character χ which gives a primitive character ψ. So for any Dirichlet character χ, $L(s,\chi)$ is the L-series of a primitive character times a finite number of factors.

For any primitive Dirichlet character $\chi \mod k$, we have the following functional equation for all s:

$$L(1-s,\chi) = \frac{k^{s-1}\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\}G(1,\chi)L(s,\bar{\chi}),$$

where $\bar{\chi}$ is the inverse of the primitive Dirichlet character χ and $G(m,\chi)$ is the Gauss sum:

$$G(m,\chi) = \sum_{r=1}^{k} \chi(r)e^{2\pi irm/k}.$$

Special Values. We give some special values of the Dirichlet L-Series.

For χ any Dirichlet character mod k and χ_1 the principal character mod k, we have

$$L(0,\chi) = -\frac{1}{k}\sum_{r=1}^{k} r\chi(r), \text{and}$$
$$L(0,\chi_1) = 0.$$

For χ a primitive character mod k with $k > 1$, we get a finite sum for $L(\chi,1)$. If $\chi(-1) = 1$, then

$$L(1,\chi) = \frac{-1}{k}G(1,\chi)\sum_{(r,m)=1, 0<r<m} \bar{\chi}(r)\ln(2\sin\frac{\pi r}{k}).$$

If $\chi(-1) = -1$, then

$$L(1,\chi) = \frac{\pi i}{k^2}G(1,\chi)\sum_{(r,m)=1, 0<r<m} \bar{\chi}(r)r.$$

Finite Fields. We define zeta functions and L-functions over finite fields F_q. Let $h = h(x) \in F_q[x]$ be monic. If h has degree d set $\mathcal{N}(h) = q^d$. Define

$$\zeta(s) = \sum_h \frac{1}{\mathcal{N}(h)^s}$$

for $s = \sigma + it$ and where the sum is over all monic polynomials in $F_q[x]$. Just as before, we get that $\zeta(s)$ converges absolutely for $\sigma > 1$ and converges uniformly for $\sigma > 1 + \delta$ where $\delta > 0$. Since q^d is the number of monic polynomials of degree d, we can rewrite this sum as

$$\zeta(s) = \sum_{d=0}^{\infty} \frac{q^d}{q^{ds}}.$$

We then see that

$$\zeta(s) = \frac{1}{1 - q^{1-s}}.$$

We also get an Euler product expansion over irreducible monic polynomials in $F_q[x]$ for $\sigma > 1$:

$$\zeta(s) = \prod_{h \text{irred.}} (1 - \mathcal{N}(h)^{-s})^{-1}.$$

Let G be the group of rational functions of monic polynomials in $F_q[x]$ and let $\bar{G}$ be a subgroup of G such that, for polynomials h_1 and h_2, if $h_1 h_2 \in \bar{G}$, then both h_1 and h_2 are also in $\bar{G}$. Let χ be a character on $\bar{G}$ and set $\chi(h) = 0$ if h is not a polynomial in $\bar{G}$. We now define (for $\sigma > 1$):

$$L(s, \chi) = \sum_{h} \chi(h) \mathcal{N}(h)^{-s}$$

over monic polynomials $h \in F_q[x]$. Then $L(s, \chi)$ converges absolutely for $\sigma > 1$ and uniformly for $\sigma > 1 + \delta$ where $\delta > 0$. We also get an Euler product expansion for $\sigma > 1$:

$$L(s, \chi) = \prod_{h \text{irred.}} (1 - \chi(h) \mathcal{N}(h)^{-s})^{-1}.$$

Bernoulli Numbers. We will now define Bernoulli numbers and polynomials and describe some of their relevant properties. We will use Dirichlet characters to generalize Bernoulli numbers and polynomials. We define the Bernoulli polynomials $B_n(x)$ by

$$F(t, x) = \frac{te^{(1+x)t}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

where F(t,x) is expanded in a formal power series. The Bernoulli numbers $B_n = B_n(0)$ are just the Bernoulli polynomials evaluated at $x = 0$:

$$F(t) = F(t,0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Since

$$F(t,x) = F(t)e^{xt} = \frac{t}{e^t - 1} e^{xt} = \left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \right),$$

we have

$$\binom{B_n(x) = \sum_{k=0}^{n} n}{k B_k x^{n-k}.}$$

Thus, $B_0(x) = 1$ and so the $B_n(x)$ are monic polynomials in $\mathbb{Q}[x]$.

References

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