Isotopy of 4-manifolds and the Seiberg-Witten equations

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Goal is to use Seiberg-Witten equations to define an invariant which will enable us to prove the following two related theorems.

Theorem 1. *There is an oriented smooth 4-manifold Z, and a diffeomorphisms $f_0, f_1 : Z \to Z$ such that $f_0 \simeq f_1$ but they are not smoothly isotopic.*

Recall that the diffeomorphisms f_0 and f_1 are *smoothly isotopic* if there is a 1-parameter family of diffeomorphisms f_t which connects them. It follows that $f_1 \circ f_0^{-1}$ is not isotopic to the identity.

To state the second result, let us denote by $\mathcal{PSC}(Y)$ the space of Riemannian metrics on Y Y with positive scalar curvature. A basic theorem in Seiberg-Witten theory states that if $b_+^2(Y) > 1$ and Y has a non-vanishing Seiberg-Witten invariant, then $\mathcal{PSC}(Y) = \emptyset$.

Theorem 2. *There is a smooth 4-manifold Z, supporting a metric of positive scalar curvature, such that $\mathcal{PSC}(Z)$ has more than one path component.*

The theorems are related: the manifold Z is the same in both cases and in fact is a connected sum of $\mathbf{CP}^2$'s and $\overline{\mathbf{CP}}^2$'s. Starting from a $\mathcal{PSC}$ metric g_0 on this connected sum, we take the metric $f_1^*(g_0)$, and show that it is not connected to g_0 by a path in $\mathcal{PSC}$.

There are analogous results in higher dimensions (P. Piazza, P. Gilkey) using index theory. The question these authors attack is somewhat harder: to detect path components (and higher homotopy groups) of the moduli space

$$\mathcal{PSC}(Z)/\mathrm{Diff}(Z)$$

Theorem 2 (and our techniques) don't seem to say anything about this harder problem.

The basic ingredient in this is 1-parameter gauge theory. (I used similar idea last year to prove a stronger version of Theorem 1, via Yang-Mills theory; this doesn't get at the positive scalar curvature result.)

Set-up: Y is a smooth oriented 4-manifold with $b^2_+ \geq 3$, and $\mathbf{W}^+ \to Y$ a Spin^c structure. Let

$$\mathcal{H} = \{(g,\mu) \in \text{Met}(Y) \times \Omega^2(Y) \mid *_g\mu = \mu\}$$

be the space of perturbations used in Seiberg-Witten theory.

For any $h \in \mathcal{H}$, we have the Seiberg-Witten equations for a connection A on $\det(W^+)$ and spinor $\psi \in \Gamma(W^+(g))$

$$F_A^+ = q(\psi) + i\mu$$
$$D_A(\phi) = 0$$

and the associated moduli space of solutions $\mathcal{M}(W^+; h)$.

For the usual Seiberg-Witten invariant, one assumes that $\dim(\mathcal{M}(W^+; h)) = 0$ and counts the solutions, with signs. We will instead assume that (formally, i.e. by an index computation)

$$\dim(\mathcal{M}(W^+; h)) = -1.$$

This assumption means that $\mathcal{M}(W^+; h)$ is empty for generic $h \in \mathcal{H}$ and requires that b^2_+ be even.

Let γ be a path in $\mathcal{H}$, and form the parameterized moduli space

$$\widetilde{\mathcal{M}}(W^+; \gamma) = \bigcup_{t \in [0,1]} \mathcal{M}(W^+; \gamma(t)).$$

For a generic path, $\widetilde{\mathcal{M}}(W^+; \gamma)$ is a smooth oriented 0-manifold, whose points we can count.

Definition: *Suppose γ is generic, and*

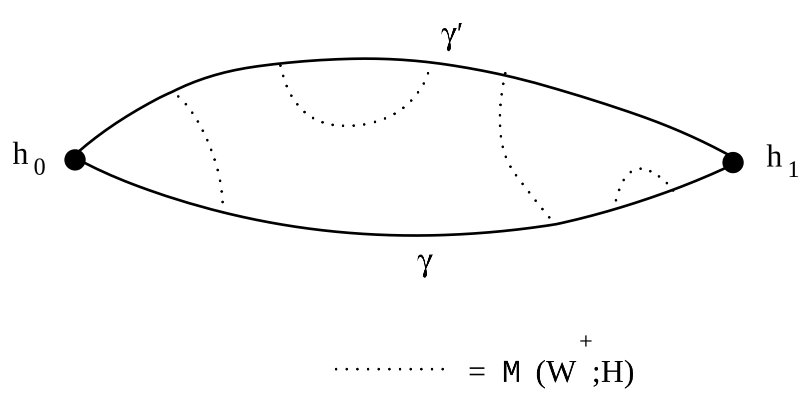
$$\mathcal{M}(W^+; \gamma(0)) = \emptyset = \mathcal{M}(W^+; \gamma(1)).$$

Then the algebraic count $\#\widetilde{\mathcal{M}}(W^+; \gamma)$ will be denoted $\mathsf{SW}(\mathbf{W}^+; \gamma)$.

Proposition 3. SW$(\mathbf{W}^+;\gamma)$ *depends only on the endpoints* $h_0 = \gamma(0)$ *and* $h_1 = \gamma(1)$.

Proof: If γ' is another path with the same endpoints, then (since $\mathcal{H}$ is contractible) there is a homotopy H from γ to γ', fixing the endpoints. The 2-parameter moduli space $\widetilde{\mathcal{M}}(\mathbf{W}^+;H)$ satisfies $\partial\widetilde{\mathcal{M}}(\mathbf{W}^+;H) = \widetilde{\mathcal{M}}(\mathbf{W}^+;\gamma') - \widetilde{\mathcal{M}}(\mathbf{W}^+;\gamma)$, from which the proposition follows. □

We will refer to SW$(\mathbf{W}^+;\gamma)$ as a 1-parameter invariant. The proof of Proposition 3 is summarized by the following picture.



Theorem 2 is based on the following observation, which is based in turn on the Weitzenböck formula.

Proposition 4. *Suppose that* $b^2_+ \geq 2$*, and that*

$$\gamma(t) = (g(t), \mu(t)) \in \mathcal{H}$$

is a path with $g(t) \in \mathcal{PSC}$ *and* $\mu(t)$ *small. Then* $\text{SW}(\mathbf{W}^+; \gamma) = 0$*.*

The relation between isotopy and the 1-parameter invariants is given by

Proposition 5. *Let* f_t *be a smooth isotopy with* $f_0 = \text{id}$*. Suppose that* h_0 *is generic and that* $h_1 = f_1^*(h_0)$*. Then* $\text{SW}(\mathbf{W}^+; \gamma) = 0$ *for any path in* $\mathcal{H}$ *connecting* h_0 *and* h_1*.*

The proof goes by applying the following lemma to the path $\gamma(t) = f_t^*(h_0)$.

Lemma 6. *Suppose* $\mathbf{W}^+$ *is a* Spin^c *structure,* $f : Y \to Y$ *is a diffeomorphism, and let* $h \in \mathcal{H}$*. Then there is a homeomorphism*

$$f^* : \mathcal{M}(\mathbf{W}^+; h) \to \mathcal{M}(f^*\mathbf{W}^+; f^*h).$$

The action of f^* *on the determinant line bundle associated to the Seiberg-Witten equations is multiplication by* $\alpha(f)\beta(f)$ *where* $\alpha(f)$ *is the spinor norm and* $\beta(f) = \pm 1$ *depending on the parity of*

$$\left(\frac{c_1(\mathbf{W}^+) - c_1(f^*\mathbf{W}^+)}{2}\right)^2.$$

Informally, $\alpha(f)\beta(f)$ measures whether f^* preserves orientation on the moduli space.

Here is an idea for a Seiberg-Witten invariant of f, which could in principle detect different isotopy classes: Start with an arbitrary (generic) $h_0 \in \mathcal{H}$, and let

$$\mathrm{SW}(\mathbf{W}^+, f) = \mathrm{SW}(\mathbf{W}^+; \gamma)$$

for a generic path γ joining h_0 to $h_1 = f^* h_0$.

Proposition 3 says that this is independent of the choice of γ, **but it is not necessarily independent of the starting point $\mathbf{h_0}$.**

There are a number of possible ways to remedy this situation, depending on the action of f on the set of (topological types of) Spin^c structures on Y. In particular, if f preserves the Spin^c structure $\mathbf{W}^+$ (e.g. if f is homotopic to the identity) the above definition will in fact turn out to be independent of h_0. However, for computations, we need the extra flexibility of allowing f to act non-trivially on the Spin^c structures.

Definition: *Let $f : Y \to Y$ be an orientation-preserving diffeomorphism, and denote by f_n the n–fold composition of f. Choose $h_0 \in \mathcal{H}$ with $\mathcal{M}(\mathbf{W}^+; h_0) = \emptyset$, and let γ be a generic path from h_0 to $f^* h_0$.*

Case 1: *Suppose that for some $n \geq 0$, we have $f_n^* \mathbf{W}^+ \approx \mathbf{W}^+$, and that n is minimal. Then we define* $\mathsf{SW}(\mathbf{W}^+, f) =$

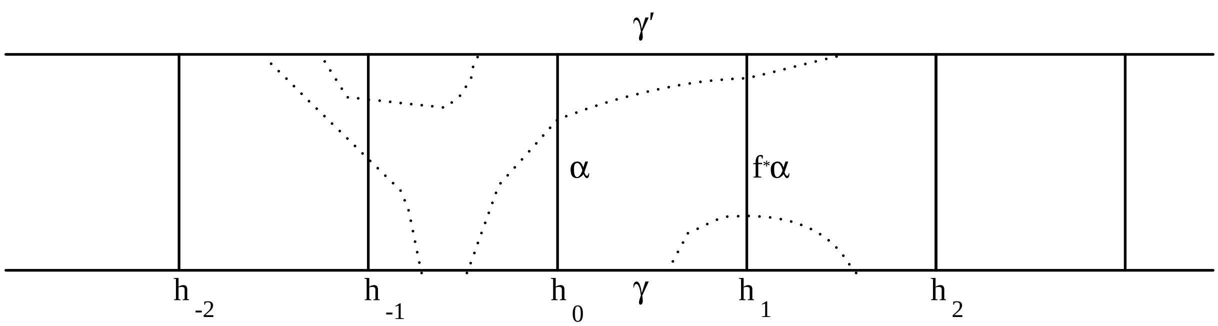
$$\sum_{j=1}^{n} \mathsf{SW}(\mathbf{W}^+; f_j^* \gamma) = \sum_{j=1}^{n} \mathsf{SW}(f_{-j}^* \mathbf{W}^+; \gamma)$$

Case 2: *If the* $\mathsf{Spin}^{\mathsf{c}}$ *structures $f_n^* \mathbf{W}^+$ are all distinct, then we define* $\mathsf{SW}(\mathbf{W}^+, f) =$

$$\sum_{j=-\infty}^{\infty} \mathsf{SW}(\mathbf{W}^+; f_j^* \gamma) = \sum_{j=-\infty}^{\infty} \mathsf{SW}(f_{-j}^* \mathbf{W}^+; \gamma)$$

The equality of the terms in these sums follows from Lemma 6. It is implicit in the definition in Case 2 that the sums are finite; this is proved by the same argument showing that the number of basic classes is finite.

The proof of independence from the initial point h_0 is a picture similar to the one which appeared in Proposition 3. To explain the proof (in case 2) let h_0, h_0' be generic points in $\mathcal{H}$, and let α be a path from h_0 to h_0'. Choose generic paths γ and γ'. Since $\mathcal{H}$ is contractible, the loop $\alpha\gamma'(f^*\alpha)^{-1}\gamma^{-1}$ can be filled in by a homotopy H. The figure shows the union of the moduli spaces $\widetilde{\mathcal{M}}(\mathbf{W}^+; f_j^* H)$.



Case 2

The proof in Case 1 is similar; the corresponding picture has finitely many boxes, and the right-most boundary is matched to the left-most boundary, using Lemma 6.

Construction of Examples

The construction of diffeomorphisms to which Theorems 1 and 2 may be applied is based on some basic facts from 4–manifold theory. Let X_0 be the connected sum

$$\#_3 \mathbf{CP}^2 \#_{19} \overline{\mathbf{CP}}^2$$

From work of Moishezon and Mandlebaum, we know that there is a complex surface X_1, homotopy equivalent to X_0, such that

$$X_0 \# \mathbf{CP}^2 \cong X_1 \# \mathbf{CP}^2.$$

(In fact, there are infinitely many such X_j, which may be constructed as logarithmic transforms of an elliptic surface.) The X_j are distinguished up to diffeomorphism by their Seiberg-Witten invariants.

Let $N = \mathbf{CP}^2 \#_3 \overline{\mathbf{CP}}^2$, so that $X_0 \# N \cong X_j \# N$ and define the manifold Z to be this connected sum.

Note that N has a large diffeomorphism group, because it contains many spheres of self-intersection ± 1 and ± 2. In particular, there is a diffeomorphism f_N of N with itself, whose action on homology (in the obvious basis for $H_2(N)$ is given by the matrix

$$\begin{pmatrix} 9 & 4 & -8 \\ 4 & 1 & -4 \\ 8 & 4 & -7 \end{pmatrix}$$

In fact, f_N is the composition of reflections in (-1)-spheres representing the homology classes $(1,1,1)$ and $(1,-1,1)$.

For each $j \geq 0$, we thus get a diffeomorphism f_j of Z (viewed as $X_j \# N$) by gluing f_N to the identity map of X_j.

The basic computation, which leads directly to Theorems 1 and 2, is proved by a gluing argument.

Theorem 7. *For an appropriately chosen* Spin^c *structure* $\mathbf{W}^+ \to Z$*, the invariant* SW$(\mathbf{W}^+; f_j)$ *is defined, and equals*

$$-2\text{SW}(\mathbf{W}^+|Z)$$

It follows that the diffeomorphisms f_j are mutually non-isotopic. Moreover, there is a positive scalar metric g_0 on Z, viewed as $X_0 \# N$, and the metrics $F_j^* g_0$ lie in different components of $\mathcal{PSC}(Z)$.

One ought to be able to deduce from these computations that π_0(Diff(Z) is infinitely generated. This is complicated because the invariant SW$(\mathbf{W}^+; f)$ does not behave well under compositions. On the other hand, we deduced this infinite generation using a similar invariant derived from the Yang-Mills equations.