Configurations of embedded spheres

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Introduction

*Joint project with Laura Starkston.*

Classical geometry studies *Real line configurations*

![Pappus configuration](image)

Pappus configuration (illustrating Pappus’ theorem)

**Important features:**

- Lines are linearly embedded $\mathbb{RP}^1$s in $\mathbb{RP}^2$; every pair meets once.
- Incidence of points and lines encoded in incidence matrix;
- Combinatorial data: specify some points of intersection (red dots in figure); there may well be more.
Typical mixed combinatorial/geometry questions: Specify number of lines and points of given multiplicity.

**Combinatorics:** Find all combinatorial configurations

**Geometry:** Find corresponding line configurations or prove non-realization.

Classically, each object in a configuration could be a

- *Real* line: copy of $\mathbb{RP}^1$, linearly embedded;
- *Complex* line: linearly embedded $\mathbb{CP}^1$ in $\mathbb{CP}^2$;
- *Real* pseudoline: copy of $\mathbb{RP}^1$, topologically embedded;

Our setting:

- *Complex* pseudoline: $S^2 \subset \mathbb{CP}^2$, homologous to $\mathbb{CP}^1$, that is
  - *topologically embedded* (locally flat)
  - *smoothly embedded*
  - *symplectically embedded.*
Relating the different notions

Arrow means configuration in one category gives configuration in another.

- \( \mathbb{R} \)-line
- \( \mathbb{C} \)-line
- \( \mathbb{R} \)-pseudoline
- \( \mathbb{C} \)-pseudolines:
  - symplectic
  - smooth
  - topological
- Combinatorial configuration
Real pseudolines

Some combinatorial configurations that aren’t geometrically realized (by straight lines) can be topologically realized (by pseudolines). An (in)famous example:

Kantor’s (1881) pseudoline configuration
Real pseudolines

A closer look—one of those lines isn’t quite straight!

But it’s still an embedded $\mathbb{RP}^1$, that is, a pseudoline.
Complex lines

Real line arrangements give complex ones by complexification: take complex solutions in $\mathbb{CP}^2$ to linear equations with real coefficients. The incidence relations are the same. This argument doesn’t work for pseudolines, but we can still complexify topologically.

**Theorem 1:** (R-Starkston 2015) Let $\mathcal{L}$ be a configuration of $\mathbb{R}$-pseudolines in $\mathbb{RP}^2$. Then there is a configuration $\mathcal{L}_C$ of $\mathbb{C}$-pseudolines in $\mathbb{CP}^2$ such that $\mathcal{L}_C \cap \mathbb{RP}^2 = \mathcal{L}$. Moreover, $\mathcal{L}_C$ is invariant under complex conjugation and has the same combinatorics as $\mathcal{L}$. 

![Diagram of complexification process](image)
Non-realizable configurations

Some combinatorial configurations are not even realized by real pseudolines. Two classic examples:

Fano Plane \((7_3)\)  
Möbius-Kantor configuration \((8_3)\)

The Fano plane is (combinatorially) the finite projective plane \(P(\mathbb{F}_2, 2)\).
Non-realizable configurations

The Möbius-Kantor configuration is realized by a configuration of complex lines (F. Levi, 1929). The Fano plane is not!

Our starting point was the following, motivated by constructions in contact and symplectic topology.

**Question:** (L. Starkston) *Is the Fano configuration realized by symplectically embedded 2-spheres in $\mathbb{CP}^2$?*

**Theorem 2:** (R-Starkston 2015) *For any prime $p$, the combinatorial configuration given by the finite projective plane $P(\mathbb{F}_{p^n}, 2)$ is not realized by a configuration of topological $\mathbb{C}$-pseudolines.*

This implies non-realization results in all of the other categories.
Theorem 2 has a basic topological mechanism. Let \( \mathcal{F} = \{F_1, \ldots, F_k\} \) be disjoint spheres in a simply-connected 4-manifold \( X \), such that \( F_i \cdot F_i = r \) with \( r \neq 0 \). Then \( \{F_1, \ldots, F_k\} \) are linearly independent in \( H_2(X; \mathbb{Z}) \).

But for a prime \( p \) dividing \( r \), there may be a \( \text{mod } p \) relation

\[
\sum_i a_i[F_i] = 0 \quad \text{in } H_2(X; \mathbb{Z}_p) \quad \text{with } a_i \in \mathbb{Z}_p. \tag{1}
\]

Each such relation corresponds to a \( p \)-fold branched covering \( \tilde{X} \to X \) with branch set a subset of \( \mathcal{F} \).

We prove Theorem 2 and related results by studying the topology of \( \tilde{X} \).
Want to show that a given combinatorial configuration is not realized.

Here’s how it works:

- Start with combinatorial configuration $C$ given by an incidence matrix.
- Assume it’s realized by a $\mathbb{C}$-pseudoline configuration $S = \{S_1, \ldots, S_k\}$ in $\mathbb{C}P^2$.
- Blow up $\mathbb{C}P^2$ at all $n$ intersection points to get $X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$.
- Let $\tilde{S} = \{\tilde{S}_1, \ldots, \tilde{S}_k\}$ be the proper transforms of the $S_i$.
- Read off relations like (1) from incidence matrix.
- Compute topological invariants of $\tilde{X}$ to get a contradiction.

More refined versions (using Heegaard Floer invariants) potentially show difference between realizing by *smooth* or *topological* $\mathbb{C}$-pseudolines.
How do the combinatorics come in?

- Mod $p$ relations amongst the surfaces $\tilde{S}_i$ form a subspace $V_C$ of the vector space $\mathbb{F}_p^k$: a $p$-ary linear code.
- Natural partial order on the vectors in $V_C$ given by $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ obtained by replacing some entries of $\mathbf{y}$ by 0’s.
- A vector $\mathbf{y} \in V_C$ is **minimal** if no nonzero $\mathbf{x}$ in $V_C$ with $\mathbf{x} \leq \mathbf{y}$.
- Vectors in $\mathbb{F}_p^n$ have a **weight**: the number of non-zero entries.
- Minimal weight vectors in $V_C$ are minimal.

Geometrically, a relation corresponding to a branched cover with branch set $\mathcal{A} \subset \tilde{S}$ is minimal if there’s no branched covering with branch set a proper subset of $\mathcal{A}$. 
An example—the Fano plane

For the Fano plane, we take $p = 2$.

The subgraph pictured below corresponds to the relation

$$\tilde{S}_2 + \tilde{S}_3 + \tilde{S}_5 + \tilde{S}_6 \equiv 0 \pmod{2}. $$

The minimal weight for this code is 4, so this is in fact minimal.
Weights and the branched cover

How does this help?

**Proposition 3:** Suppose that $X$ is simply-connected, and that $\tilde{X} \to X$ is a $p$-fold cyclic branched cover with branch set $\mathcal{F} = \{F_1, \ldots, F_k\}$. If the corresponding vector is minimal, then $H_1(\tilde{X}; \mathbb{C}) = 0$.

From Proposition 3 and traditional methods (Euler characteristic, G-signature theorem), we can determine

- $H_2(\tilde{X}; \mathbb{C})$, along with the action of $\mathbb{Z}_p$.
- The (equivariant) intersection form on $\tilde{X}$.

**Summary:** Combinatorial information leads to information about the topology of $\tilde{X}$. 
For the Fano plane, we are looking at $X = \mathbb{CP}^2$ blown up at 7 points. For the branched cover corresponding to the weight 4 relation, we find that

- $b_2(\tilde{X}) = 10$
- The signature of $\tilde{X}$ is $-8$.
- From the geometry, we can construct 10 disjoint spheres in $\tilde{X}$, each with negative self-intersection.

This is a contradiction, so the Fano configuration is not realized, even by locally flat 2-spheres.
Finding the spheres

Separate spheres by blowing up at the points $Q_i$. 

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Finding the spheres: proper transforms of the $S_i$ in $X$
Finding 10 spheres in the double cover $\tilde{\mathcal{X}}$

\[ S_1'' \cdot S_1 = S_4' \cdot S_4 = S_7' \cdot S_7 = -2 \]

\[ S_1' \cdot S_1' = S_4' \cdot S_4' = S_7' \cdot S_7' = -2 \]

\[ S_2' \cdot S_2' = S_3' \cdot S_3' = S_5' \cdot S_5' = S_6' \cdot S_6' = -1 \]
Sketch of complexification Theorem 1

Represent $\mathbb{R}$-pseudoline configuration by a ‘wiring diagram’, then do this...
Sketch of complexification Theorem 1

... and plug it into this