Applications of 1-parameter gauge theory

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1. Introduction

Goal is to demonstrate a simple technique, useful in a variety of geometric and topological problems about 4-manifolds. Here are two typical sorts of applications that I will discuss.

- Distinguishing diffeomorphisms up to isotopy; equivalently detecting components of the diffeomorphism group, \( \text{Diff}(X) \) of a 4-manifold \( X \).

- Distinguishing components of the space, \( \text{PSC}(X) \) of metrics of positive curvature on \( X \).

In principle, similar methods work to understand higher homotopy groups of these spaces. Other people (P. Seidel, P. Kronheimer) have used similar methods to study the topology of the group \( \text{Symp}(X, \omega) \) of automorphisms of a symplectic manifold.
To describe the technique, look at an old and familiar result: the Jordan Curve Theorem, for smooth curves in the plane. (The case of continuous curves is harder to treat directly.) So, let $C$ be a closed smooth, oriented connected curve in $\mathbb{R}^2$. The Jordan Curve Theorem says that $C$ separates $\mathbb{R}^2$ into two components.

One proof of this proceeds by defining an invariant $I(x_0, x_1) \in \mathbb{Z}$ for a pair of points in $\mathbb{R}^2 - C$. Choose a path $\gamma : I \rightarrow \mathbb{R}^2$ joining $x_0$ to $x_1$; assume that it cuts $C$ transversally wherever they intersect. To each intersection point, assign a number $\pm 1$ depending on whether the tangent vector to $C$, followed by the tangent to $\gamma$, is a positively oriented basis of $\mathbb{R}^2$. Add up all of the numbers to get an invariant

$$I(x_0, x_1) = \#(C \cap \gamma)$$

where the $\#$ sign indicates that we are making a signed count as described.

The procedure is illustrated by the following figure.
The figure illustrates an important point: there are lots of paths from $x_0$ to $x_1$. Why do these give the same count for $I(x_0, x_1)$?
The key to the answer is that the plane is simply connected. If we had two curves, say $\gamma_0$ and $\gamma_1$, we can deform one to the other, keeping the endpoints fixed. During the deformation, intersection points with $C$ are either annihilated in pairs, or created in pairs. The signs in these pairs are always opposite, so $I(x_0, x_1)$ is well-defined.
The independence of $I(x_0, x_1)$ from the choice of path proves that $\mathbb{R}^2 - C$ is disconnected. For it is not hard to find two points on opposite sides of $C$ (this makes sense locally). These have $I(x_0, x_1) = \pm 1$ and so there is no curve joining them that misses $C$.

2. The Seiberg-Witten Equations

To apply this scheme to the geometric problems mentioned above, we will need the Seiberg-Witten equations. The data needed to write down the Seiberg-Witten equations are:

- A smooth, oriented 4-manifold $X$ together with a Riemannian metric $g$.

- A spin$^c$ structure on $X$. This consists of two complex 2-plane bundles $W^\pm \to X$, together with a “Clifford multiplication” $c : T^*X \times W^\pm \to W^{\mp}$.

- A closed 2-form $\mu$. 
The metric determines the $\ast$-operator on 2-forms, written $\ast : \Omega^2(X) \to \Omega^2(X)$. With respect to a local orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the cotangent space, it is given by

$$e_1 \wedge e_2 \to e_3 \wedge e_4, \quad e_1 \wedge e_3 \to -e_2 \wedge e_4, \quad \text{etc.}$$

The $\ast$-operator is an involution, so the forms split into $\pm 1$ eigenspaces:

$$\Omega^2 \cong \Omega^2_+ \oplus \Omega^2_-$$

The variables in the Seiberg-Witten equations are a (spin) connection $A$ on the bundle $W^+$ and a section $\psi$ of that bundle. The connection is a ‘directional derivative’, and is really an operator

$$\nabla_A : \Gamma(W^+) \to \Gamma(T^* X \otimes W^+)$$

It is required to satisfy a product rule (Leibniz rule) with respect to the Clifford multiplication. Composing $\nabla_A$ with Clifford multiplication gives the Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$.
Finally, the connection has its curvature, which in this situation is a 2-form $F_A$. Using the $\ast$-operator, the curvature splits as $F_A^+ + F_A^-$. The Seiberg-Witten equations (for $A$, $\psi$) are:

$$D_A\psi = 0$$

$$c(F_A^+ + \mu^+) = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2 \text{Id}_{W^+}$$

Both sides of the second equation are automorphisms of the bundle $W^+$. 
The solutions to these equations, divided by an appropriate symmetry group, form the moduli space $\mathcal{M}_{g,\mu}$. We are interested in how these moduli spaces change as you vary the parameters $(g, \mu)$ in the space $\mathcal{P}$ of all metrics and closed 2-forms. This is a contractible space, and will play the role of the plane $\mathbb{R}^2$ in the Jordan Curve Theorem. For simplicity, we will often denote the pair $(g, \mu) \in \mathcal{P}$ by a single letter $h$.

For a generic choice of $h \in \mathcal{P}$, the moduli space $\mathcal{M}_h$ will be an oriented compact smooth manifold of dimension $d$ given by the topological formula

$$d = \frac{c_1^2(W^+) - (2\chi(X) + 3\sigma(X))}{4}$$
Usually, one wants to arrange things so that $d = 0$, in which case the moduli space is a set of points which may be counted with signs to give the Seiberg-Witten invariant of the 4-manifold. It can be shown that this count is independent of the parameter $h$. In our applications, however, we will arrange the topology of $X$ so that $d = -1$.

This seems like a very silly thing to do: $d$ being negative means that for generic $h \in \mathcal{P}$, the moduli space is empty! (Just as a generic point in the plane will not lie on a curve.) Instead of getting an invariant of $X$, we get the following invariant of a pair of parameters $h_0, h_1 \in \mathcal{P}$. Choose a generic path $h : [0, 1] \to \mathcal{P}$ with $h(0) = h_0$ and $h(1) = h_1$. Then form the 1-parameter moduli space

$$
\tilde{\mathcal{M}}_h = \bigcup_{t \in [0,1]} \mathcal{M}_{h(t)}
$$
This moduli space is 0-dimensional, and so we can count its points (as always, with signs) to get the 1-parameter invariant

\[ I(h_0, h_1) = \#\tilde{\mathcal{M}}_h \]

Just as in our sketch of the Jordan Curve theorem, this algebraic count is independent of the path. In a deformation of paths (leaving endpoints fixed) the only thing that can happen is creation/annihilation of pairs of points having opposite signs. A similar idea works to define \( k \)-parameter invariants, in the setting where the ‘dimension’ \( d = -k \).
3. Positive scalar curvature metrics

Our first application of the 1-parameter invariants is to the rough classification of metrics of positive curvature. Recall that associated to a Riemannian metric $g$ on a manifold $X$ is an assortment of curvatures, the simplest of which is the scalar curvature. This is a function $s_g : X \to \mathbb{R}$.

What kinds of functions on $X$ could be $s_g$ for some metric $g$? In particular, it is known that there are topological obstructions to $X$ having a metric for which $s_g$ is always positive. For instance, among surfaces, only the 2-sphere has such metrics, by the Gauss-Bonnet theorem. If there is such a metric on $X$, we can try to study the set of such metrics, which form a space $\text{PSC}(X)$. 

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If $\text{PSC}(X)$ is non-empty, it is a large space—it is open in the (infinite dimensional) space of Riemannian metrics. We can still ask questions about its topology, for instance whether the space $\text{PSC}(X)$ is connected. There are results known about this question when $X$ has dimension at least 5, but previously no one could say anything about dimension 4. (Dimensions 2 and 3 are special, for other reasons.)

**Theorem 1.** *There are simply-connected 4-manifolds, for which $\text{PSC}(X)$ has infinitely many components.*

The manifolds in question are not very complicated; they are built by gluing together copies of the complex projective plane $\mathbb{CP}^2$. Such manifolds have PSC metrics gotten by gluing up the standard (Fubini-Study) metric on $\mathbb{CP}^2$. 
To detect the components of $\text{PSC}(X)$, we use the following principle, observed by Witten. If $h = (g, \mu)$ where $s_g > 0$ and $\mu$ is small, then $\mathcal{M}_h$ is empty. This implies:

**Lemma 2.** Suppose that $g_0, g_1$ have positive scalar curvature, and lie in the same path component of $\text{PSC}(X)$. Then (for sufficiently small $\mu$), we have that $I(h_0, h_1) = 0$.

The theorem is proved by finding a second metric $g_1$ with $I(h_0, h_1) \neq 0$. This isn’t so easy! The metric $g_1$ is of the form $f^* g_0$, where $f$ is a certain diffeomorphism of $X$. 
4. Topology of the diffeomorphism group

Ideally, given a smooth manifold $X^n$, one would like to understand all of its automorphisms. This is too complicated, so instead, topologists settle for understanding what we can about the topology of the group, $\text{Diff}(X)$, of diffeomorphisms of $X$. In particular, we study the homotopy groups $\pi_k(\text{Diff}(X))$, the simplest of which is $\pi_0$, the group of path components of $\text{Diff}(X)$.

Two diffeomorphisms that are connected in the diffeomorphism group are said to be isotopic. This turns out to be a hard relation to study directly, so in high dimensions one studies instead the relation of pseudo-isotopy, which is easier to deal with.
**Definition:** Diffeomorphisms \( f_0, f_1 : X \to X \) are pseudo-isotopic if there is a diffeomorphism \( F : X \times I \to X \times I \) such that \( F(x, 0) = f_0(x) \) and \( F(x, 1) = f_1(x) \).

A pseudo-isotopy gives an isotopy if \( F \) preserves levels, i.e., \( F : X \times \{t\} \xrightarrow{\cong} X \times \{t\} \). A famous theorem of Jean Cerf (1970) says that if the dimension \( n \) is at least 5 and \( X \) is simply connected, then pseudo-isotopy implies isotopy. There are more complicated versions if \( \pi_1(X) \neq \{1\} \), but the restriction on dimension is essential.

**Theorem 3.** There is a simply connected 4-manifold \( X \), and diffeomorphism \( f : X \to X \) that is pseudo-isotopic, but not isotopic to the identity.

In fact, the subgroup of \( \pi_0(\text{Diff}(X)) \) given by such diffeomorphisms is infinitely generated.
This theorem is proved using 1-parameter gauge theory. The manifold $X$ is chosen so that for an appropriate spin$^c$ structure, the moduli space has dimension $d = -1$. Choose a generic $h_0 \in \mathcal{P}$, and consider its pull-back $f^*h_0$ by the diffeomorphism. Since $f^*h_0 \in \mathcal{P}$, we can hope to define an invariant of $f$ by

$$I(f) = I(h_0, f^*h_0)$$

The first point is that this is independent of the starting point $h_0$; we already know it's independent of the path. The second point is to prove that it is an invariant of the isotopy class of $f$; this is fairly straightforward. The hard part is to actually calculate the invariant for some examples.

Finally, the two applications are related; the PSC metrics in the first application are gotten by pulling back a standard metric using these ‘exotic’ diffeomorphisms.