Applications of 1-parameter gauge theory

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1. Introduction

Goal is to demonstrate a simple technique, useful in a variety of geometric and topological problems about 4-manifolds. Here are two typical sorts of applications that I will discuss.

- Distinguishing diffeomorphisms up to isotopy; equivalently detecting components of the diffeomorphism group, Diff(X) of a 4-manifold X.
- Distinguishing components of the space, PSC(X) of metrics of positive curvature on X.

In principle, similar methods work to understand higher homotopy groups of these spaces. Other people (P. Seidel, P. Kronheimer) have used similar methods to study the topology of the group Symp(X, ω) of automorphisms of a symplectic manifold.

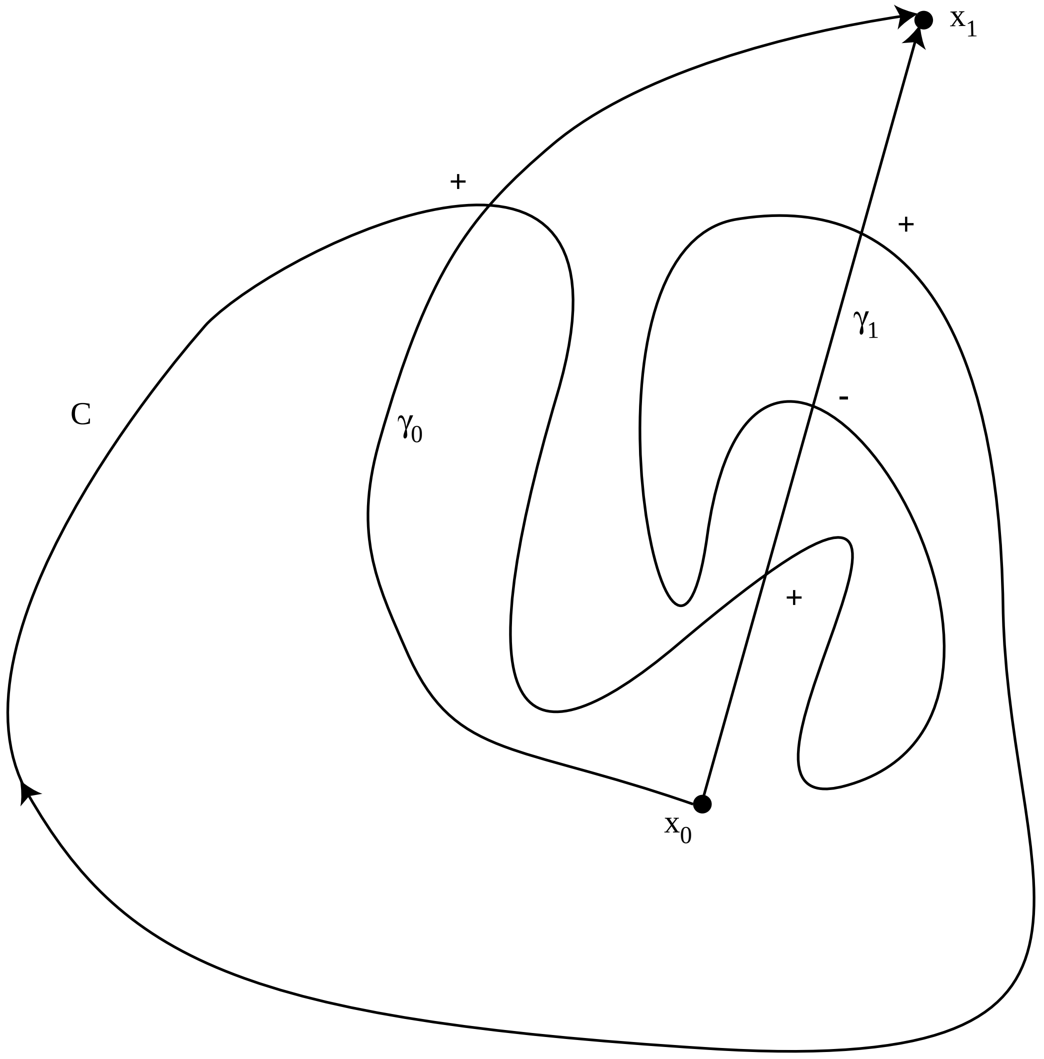
To describe the technique, look at an old and familiar result: the Jordan Curve Theorem, for smooth curves in the plane. (The case of continuous curves is harder to treat directly.) So, let C be a closed smooth, oriented connected curve in $\mathbf{R}^2$. The Jordan Curve Theorem says that C separates $\mathbf{R}^2$ into two components.

One proof of this proceeds by defining an invariant $I(x_0, x_1) \in \mathbf{Z}$ for a pair of points in $\mathbf{R}^2 - C$. Choose a path $\gamma : I \to \mathbf{R}^2$ joining x_0 to x_1; assume that it cuts C transversally wherever they intersect. To each intersection point, assign a number ±1 depending on whether the tangent vector to C, followed by the tangent to γ, is a positively oriented basis of $\mathbf{R}^2$. Add up all of the numbers to get an invariant

$$I(x_0, x_1) = \#(C \cap \gamma)$$

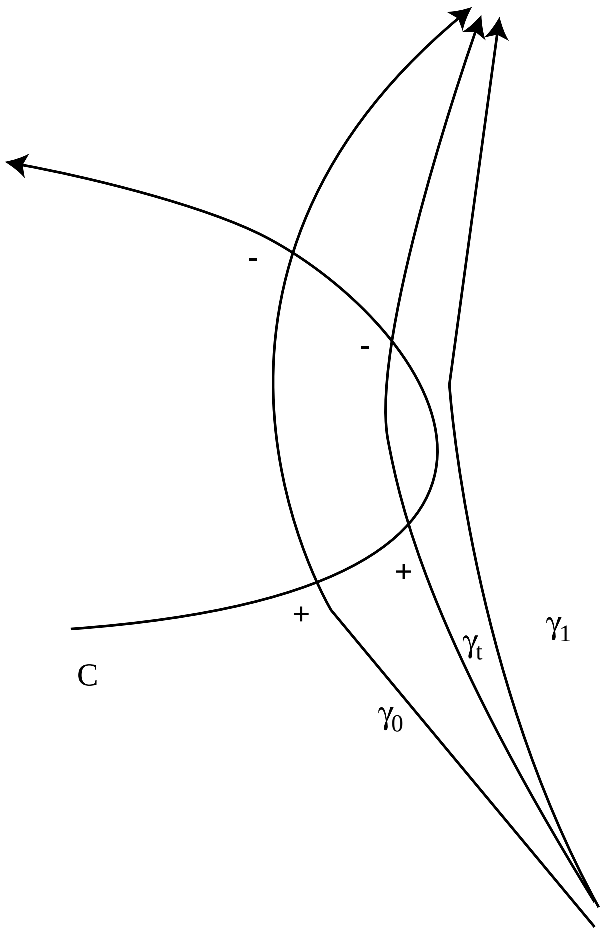
where the # sign indicates that we are making a signed count as described.

The procedure is illustrated by the following figure.



The figure illustrates an important point: there are lots of paths from x_0 to x_1. Why do these give the same count for $I(x_0, x_1)$?

The key to the answer is that the plane is simply connected. If we had two curves, say γ_0 and γ_1, we can deform one to the other, keeping the endpoints fixed. During the deformation, intersection points with C are either annihilated in pairs, or created in pairs. The signs in these pairs are always opposite, so $I(x_0, x_1)$ is well-defined.



The independence of $I(x_0, x_1)$ from the choice of path proves that $\mathbf{R}^2 - C$ is disconnected. For it is not hard to find two points on opposite sides of C (this makes sense locally). These have $I(x_0, x_1) = \pm 1$ and so there is no curve joining them that misses C.

2. The Seiberg-Witten Equations

To apply this scheme to the geometric problems mentioned above, we will need the Seiberg-Witten equations. The data needed to write down the Seiberg-Witten equations are:

- A smooth, oriented 4-manifold X together with a Riemannian metric g.
- A spin^c structure on X. This consists of two complex 2-plane bundles $W^\pm \to X$, together with a "Clifford multiplication" $c : T^*X \times W^\pm \to W^\mp$.
- A closed 2-form μ.

The metric determines the $*$-operator on 2-forms, written $* : \Omega^2(X) \to \Omega^2(X)$. With respect to a local orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the cotangent space, it is given by

$$e_1 \wedge e_2 \to e_3 \wedge e_4, \quad e_1 \wedge e_3 \to -e_2 \wedge e_4, \quad \text{etc.}$$

The $*$-operator is an involution, so the forms split into ± 1 eigenspaces:

$$\Omega^2 \cong \Omega^2_+ \oplus \Omega^2_-$$

The variables in the Seiberg-Witten equations are a (spin) connection A on the bundle W^+ and a section ψ of that bundle. The connection is a 'directional derivative', and is really an operator

$$\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$$

It is required to satisfy a product rule (Leibniz rule) with respect to the Clifford multiplication. Composing ∇_A with Clifford multiplication gives the *Dirac operator* $D_A : \Gamma(W^+) \to \Gamma(W^-)$.

Finally, the connection has its curvature, which in this situation is a 2-form F_A. Using the $*$-operator, the curvature splits as $F_A^+ + F_A^-$.

The Seiberg-Witten equations (for A, ψ) are:

$$D_A\psi = 0$$
$$c(F_A^+ + \mu^+) = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2 \, \mathrm{Id}_{W^+}$$

Both sides of the second equation are automorphisms of the bundle W^+.

The solutions to these equations, divided by an appropriate symmetry group, form the moduli space $\mathcal{M}_{g,\mu}$. We are interested in how these moduli spaces change as you vary the parameters (g, μ) in the space $\mathcal{P}$ of all metrics and closed 2-forms. This is a contractible space, and will play the role of the plane $\mathbf{R}^2$ in the Jordan Curve Theorem. For simplicity, we will often denote the pair $(g, \mu) \in \mathcal{P}$ by a single letter h.

For a generic choice of $h \in \mathcal{P}$, the moduli space $\mathcal{M}_h$ will be an oriented compact smooth manifold of dimension d given by the topological formula

$$d = \frac{c_1^2(W^+) - (2\chi(X) + 3\sigma(X))}{4}$$

Usually, one wants to arrange things so that $d = 0$, in which case the moduli space is a set of points which may be counted with signs to give the Seiberg-Witten invariant of the 4-manifold. It can be shown that this count is independent of the parameter h. In our applications, however, we will arrange the topology of X so that $d = -1$.

This seems like a very silly thing to do: d being negative means that for generic $h \in \mathcal{P}$, the moduli space is empty! (Just as a generic point in the plane will not lie on a curve.) Instead of getting an invariant of X, we get the following invariant of a *pair* of parameters $h_0, h_1 \in \mathcal{P}$. Choose a generic path $h : [0, 1] \to \mathcal{P}$ with $h(0) = h_0$ and $h(1) = h_1$. Then form the *1-parameter moduli space*

$$\tilde{\mathcal{M}}_h = \cup_{t \in [0,1]} \mathcal{M}_{h(t)}$$

This moduli space is 0-dimensional, and so we can count its points (as always, with signs) to get the 1-parameter invariant

$$I(h_0, h_1) = \#\tilde{\mathcal{M}}_h$$

Just as in our sketch of the Jordan Curve theorem, this algebraic count is independent of the path. In a deformation of paths (leaving endpoints fixed) the only thing that can happen is creation/annihilation of pairs of points having opposite signs. A similar idea works to define k-parameter invariants, in the setting where the 'dimension' $d = -k$.

3. Positive scalar curvature metrics

Our first application of the 1-parameter invariants is to the rough classification of metrics of positive curvature. Recall that associated to a Riemannian metric g on a manifold X is an assortment of curvatures, the simplest of which is the scalar curvature. This is a function $s_g : X \to \mathbf{R}$.

What kinds of functions on X could be s_g for some metric g? In particular, it is known that there are topological obstructions to X having a metric for which s_g is always positive. For instance, among surfaces, only the 2-sphere has such metrics, by the Gauss-Bonnet theorem. If there is such a metric on X, we can try to study the set of such metrics, which form a space PSC(X).

If PSC(X) is non-empty, it is a large space–it is open in the (infinite dimensional) space of Riemannian metrics. We can still ask questions about its topology, for instance whether the space PSC(X) is connected. There are results known about this question when X has dimension at least 5, but previously no one could say anything about dimension 4. (Dimensions 2 and 3 are special, for other reasons.)

Theorem 1. *There are simply-connected* 4-*manifolds, for which* PSC(X) *has infinitely many components.*

The manifolds in question are not very complicated; they are built by gluing together copies of the complex projective plane $\mathbf{CP}^2$. Such manifolds have PSC metrics gotten by gluing up the standard (Fubini-Study) metric on $\mathbf{CP}^2$.

To detect the components of PSC(X), we use the following principle, observed by Witten. If $h = (g, \mu)$ where $s_g > 0$ and μ is small, then $\mathcal{M}_h$ is empty. This implies:

Lemma 2. *Suppose that g_0, g_1 have positive scalar curvature, and lie in the same path component of* PSC(X)*. Then (for sufficiently small μ_i), we have that $I(h_0, h_1) = 0$.*

The theorem is proved by finding a second metric g_1 with $I(h_0, h_1) \neq 0$. This isn't so easy! The metric g_1 is of the form $f^* g_0$, where f is a certain diffeomorphism of X.

4. Topology of the diffeomorphism group

Ideally, given a smooth manifold X^n, one would like to understand all of its automorphisms. This is too complicated, so instead, topologists settle for understanding what we can about the topology of the group, Diff(X), of diffeomorphisms of X. In particular, we study the homotopy groups π_k(Diff(X)), the simplest of which is π_0, the group of path components of Diff(X).

Two diffeomorphisms that are connected in the diffeomorphism group are said to be *isotopic*. This turns out to be a hard relation to study directly, so in high dimensions one studies instead the relation of *pseudo-isotopy*, which is easier to deal with.

Definition: *Diffeomorphisms* $f_0, f_1 : X \to X$ *are pseudo-isotopic if there is a diffeomorphism* $F : X \times I \to X \times I$ *such that* $F(x, 0) = f_0(x)$ *and* $F(x, 1) = f_1(x)$*.*

A pseudo-isotopy gives an isotopy if F preserves levels, i.e., $F : X \times \{t\} \overset{\cong}{\to} X \times \{t\}$. A famous theorem of Jean Cerf (1970) says that if the dimension n is at least 5 and X is simply connected, then pseudo-isotopy implies isotopy. There are more complicated versions if $\pi_1(X) \neq \{1\}$, but the restriction on dimension is essential.

Theorem 3. *There is a simply connected* 4-*manifold* X*, and diffeomorphism* $f : X \to X$ *that is pseudo-isotopic, but not isotopic to the identity.*

In fact, the subgroup of $\pi_0(\mathrm{Diff}(X))$ given by such diffeomorphisms is infinitely generated.

This theorem is proved using 1-parameter gauge theory. The manifold X is chosen so that for an appropriate spin^c structure, the moduli space has dimension $d = -1$. Choose a generic $h_0 \in \mathcal{P}$, and consider its pull-back f^*h_0 by the diffeomorphism. Since $f^*h_0 \in \mathcal{P}$, we can hope to define an invariant of f by

$$I(f) = I(h_0, f^*h_0)$$

The first point is that this is independent of the starting point h_0; we already know it's independent of the path. The second point is to prove that it is an invariant of the isotopy class of f; this is fairly straightforward. The hard part is to actually calculate the invariant for some examples.

Finally, the two applications are related; the PSC metrics in the first application are gotten by pulling back a standard metric using these 'exotic' diffeomorphisms.