Rohlin's Invariant and 4-dimensional Gauge Theory

Daniel Ruberman
Nikolai Saveliev

Overall theme: relation between Rohlin-type invariants and gauge theory, especially in dimension 4.

Background: Recall Rohlin's theorem: the signature $\sigma(X)$ of a closed, smooth, spin 4-manifold X is divisible by 16. This leads to an invariant of a spin 3-manifold (M, s). Write

$$(M, s) = \partial(W, s)$$

where W is smooth and the spin structure extends, and define

$$\rho(M, s) = \frac{\sigma(W)}{8} \pmod{2\mathbf{Z}} \in \mathbf{Q}/2\mathbf{Z}$$

If M is a homology sphere, then this lives in $\mathbf{Z}/2\mathbf{Z} = \mathbf{Z}_2$ as usual. $\rho(M)$ is a homology-cobordism invariant, giving a homomorphism $\rho : \Theta_3^H \to \mathbf{Z}_2$.

A 4-dimensional Rohlin invariant:

Let X be an oriented homology $S^1 \times S^3$, with a choice of generator $\alpha \in H_3(X; \mathbf{Z})$. Choose a 3-manifold $M^3 \subset X$ carrying α. Give X a spin structure; then M inherits a spin structure s.

Definition: $\rho(X) = \rho(M, s)$.

This is independent of choice of spin structure on X, and of the submanifold M. In contrast to the homology sphere case, it's not necessary that $\rho(X) \in \mathbf{Z}/2\mathbf{Z}$. For the rest of the talk, we will assume that X is a $\mathbf{Z}[\mathbf{Z}]$-homology $S^1 \times S^3$, i.e. that its universal abelian cover has the homology of S^3. In this case, $\rho(X) \in \mathbf{Z}/2\mathbf{Z}$.

Basic questions:

Question 1: Is there a homotopy sphere Σ with $\rho(\Sigma) \neq 0$?

Answer: No! (Casson 1985)

Question 2: Is there a homotopy $S^1 \times S^3$ with $\rho \neq 0$?

Answer: ?????

Related questions about homology cobordism:

Question 3: If $\rho(\Sigma) \neq 0$, can $\Sigma \# -\Sigma$ bound a contractible manifold?

Question 4: Is there an homology sphere Σ with $\rho(\Sigma) \neq 0$ that is of order 2 in Θ_3^H?

Brief recollection of Casson's invariant:

Casson defined, for an oriented homology sphere Σ, an integer invariant $\lambda(\Sigma)$. Important properties:

- $\lambda(\Sigma) = 0$ if Σ is a homotopy sphere.
- $\lambda(\Sigma) \equiv \rho(\Sigma) \pmod 2$

These properties obviously imply a negative answer to Question 1.

Notation:

- $P \to \Sigma$ is a trivial SU(2) bundle.
- $\mathcal{R}^*(\Sigma)$ = irreducible flat connections on P = irreducible SU(2) representations of $\pi_1(\Sigma)$.
- Each $\alpha \in \mathcal{R}^*(\Sigma)$ has a sign $\epsilon(\alpha) = \pm 1$ determined by spectral flow.

If $\mathcal{R}^*(\Sigma)$ is a smooth 0-dimensional manifold, then (Taubes' version of) Casson invariant is (up to an overall sign):

$$\lambda(\Sigma) = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^*(\Sigma)} \epsilon(\alpha) = \frac{1}{2} \chi(HF_*(\Sigma))$$

Here $HF_*(\Sigma)$ denotes the *Floer Homology*.

If $\mathcal{R}^*(\Sigma)$ is not smooth, then have to use perturbed equations $F_\alpha + \nabla h(\alpha) = 0$, with moduli space $\mathcal{R}^*_h(\Sigma)$. If $H_1(\Sigma) \neq 0$, then there are versions of Casson's invariant counting projectively flat connections on odd U(2) bundles.

An approach in dimension 4

To find a homotopy $S^1 \times S^3$ with $\rho \neq 0$, find a homology sphere Σ with $\rho(\Sigma) \neq 0$, and with $\Sigma \cup -\Sigma$ the boundary of a contractible 4-manifold. Nobody has a clue how to do this.

To show that $\rho(X) = 0$ for X a homotopy $S^1 \times S^3$, try to mimic Casson's approach: find an invariant $\lambda(X) \in \mathbf{Z}$ with

- $\lambda(X) = 0$ if X is a homotopy $S^1 \times S^3$.
- $\lambda(X) \equiv \rho(X) \pmod 2$

A good candidate for such an invariant already exists, due to M. Furuta/H. Ohta (1993).

For $P \to X$, the trivial SU(2) bundle, the anti-self-dual (ASD) connections are precisely the flat connections. The moduli space splits into reducible and irreducible parts:

$$\mathcal{M}(X) = \mathcal{M}^{\mathsf{red}} \coprod \mathcal{M}^*$$

Assume (after perturbation if necessary) that $\mathcal{M}^*(X)$ is smooth, so it's an oriented 0-manifold. Each element $\alpha \in \mathcal{M}^*(X)$ gets a sign $\delta(\alpha) = \pm 1$ and Furuta and Ohta define

$$\lambda_{\mathsf{FO}}(X) = \frac{1}{4} \sum_{\alpha \in \mathcal{M}^*(X)} \delta(\alpha)$$

Proposition 1. $\lambda_{FO}(X)$ *is independent of metric and perturbations, and so is well-defined.*

The hypothesis on the infinite cyclic cover means that $\mathcal{M}^{\mathsf{red}}$ remains isolated from $\mathcal{M}^*$ even in a 1-parameter variation of metrics/perturbations. Except for this, the theory is basically that of Donaldson invariants.

Conjecture: *If X is a $\mathbf{Z}[\mathbf{Z}]$-homology $S^1 \times S^3$, then*

1. $\lambda_{FO}(X) \in \mathbf{Z}$. (Not obvious because of the 1/4 *factor in the definition.)*

2. $\lambda_{FO}(X) \equiv \rho(X) \pmod 2$.

3. If $f : X \xrightarrow{\cong} X$ reverses orientation but is the identity on $H_1(X)$, then $\lambda_{FO}(X) = 0$.

Parts 1 and 2 would imply that $\rho(X)$ vanishes if X is a homotopy $S^1 \times S^3$; adding in part 3 would answer the question about order 2 elements in Θ_3^H.

Furuta-Ohta invariant of mapping tori.

To test these conjectures, we evaluated $\lambda_{\mathrm{FO}}(X)$ when X fibers over S^1, ie $X = S^1 \times_\tau \Sigma^3$. Get answers under one of two assumptions:

- *$\tau : \Sigma \to \Sigma$ has finite order.*
- *The character variety $\mathcal{R}^*(\Sigma)$ is smooth and of dimension* 0*.*

Comparing the two answers in the case when both of these hold gives some surprising results.

In the case when τ has finite order, we express $\lambda_{\mathsf{FO}}(S^1 \times_\tau \Sigma)$ in terms of the *equivariant Casson invariant* of Σ. The monodromy τ induces a map $\tau^* : \mathcal{R}^*(\Sigma) \to \mathcal{R}^*(\Sigma)$, and we denote by $\mathcal{R}^\tau$ its fixed point set. It can be shown that there exist τ-equivariant perturbations h such that $\mathcal{R}^\tau_h$ is smooth. For simplicity, will generally suppress the perturbation in the notation.

Remark: We are only concerned with the fixed points; it may happen that there is no equivariant perturbation making the full space $\mathcal{R}^*_h$ smooth.

Each element $\alpha \in \mathcal{R}^\tau$ gets a sign $\epsilon^\tau(\alpha)$, determined by the spectral flow of operators

$$\begin{pmatrix} 0 & d_\alpha^* \\ d_\alpha & -* d_\alpha \end{pmatrix}$$

on equivariant differential forms. This may well differ from the non-equivariant sign $\epsilon(\alpha)$.

Definition: *The equivariant Casson invariant is given by* $\lambda^\tau(\Sigma) = \frac{1}{2}\sum_{\alpha\in\mathcal{R}^\tau} \epsilon^\tau(\alpha)$. *It is independent of choices of perturbations and metrics.*

Theorem 2. *If* τ *has finite order, then*

$$\lambda_{FO}(S^1 \times_\tau \Sigma) = \lambda^\tau(\Sigma)$$

The first step in proving the theorem is to find τ-equivariant perturbations. These are compatible with perturbations of the ASD equations, so that restriction gives a map from $\mathcal{M}_h(S^1 \times_\tau \Sigma) \to \mathcal{R}^\tau(\Sigma)$. Because of the choice of monodromy in the S^1 direction, this map is 2–1. The other main step is to show that the sign with which an ASD connection contributes to λ_{FO} is given by equivariant spectral flow. This uses techniques from Atiyah-Patodi-Singer.

The general case, where τ has infinite order, is more complicated because we do not have equivariant perturbations. However, if we are lucky, and no perturbation is needed, we get a nice result, reminiscent of the Lefschetz fixed point theorem.

Theorem 3. *Suppose that $\mathcal{R}^*(\Sigma)$ is smooth and 0-dimensional. Then $\lambda_{FO}(S^1 \times_\tau \Sigma) = \frac{1}{2}L(\tau_*)$, where $L(\tau_*)$ denotes the Lefschetz number of the action of τ on the Floer homology.*

The proof of this is a fairly formal deduction from some gluing theorems in Floer homology, using the Hopf trace formula.

The minimal hypothesis to make this theorem work is that there is a perturbation h that is equivariant with respect to τ, and such that the whole moduli space $\mathcal{R}^*_h$ is smooth. This is much stronger than the hypothesis for theorem 2. We conjecture that the result of theorem 3 holds without any hypothesis about equivariant perturbations.

In examples, it is much harder to compute the Lefschetz number directly than it is to compute the equivariant Casson invariants.

The theory has a rather different character depending on whether τ has fixed points or not. If there are fixed points, then $\Sigma' = \Sigma/\mathbf{Z}_n$ is a homology sphere, and we can make use of some formulas of Collin-Saveliev (2001).

If τ acts freely, then Σ' is a $\mathbf{Z}[\mathbf{Z}_n]$-homology lens space, and $\lambda^\tau(\Sigma)$ is a sum of invariants of a type discussed by Boyer-Nicas (1990) and Boyer-Lines (1990). These invariants count *irreducible* flat connections on Σ'; their gauge theoretic definition is implicit in work of Cappell-Lee-Miller (1999). If n is odd, only the Boyer-Nicas invariant appears.

Back to the Rohlin invariant

In the case that τ has finite order n, we can use theorem 2 to verify the conjectured relationship between λ_{FO} and the Rohlin invariant. In the case that τ has fixed points, Collin-Saveliev showed that

$$\lambda^\tau(\Sigma) = n \cdot \lambda(\Sigma') + \frac{1}{8} \sum_{m=0}^{n-1} \sigma_{m/n}(k),$$

where $\sigma_\alpha(k)$ is the Tristram-Levine knot signature. From this it follows that $\lambda^\tau(\Sigma) \equiv \rho(\Sigma) \equiv \rho(S^1 \times_\tau \Sigma) \pmod 2$. Thus the conjecture holds in this case.

In case that τ acts freely, then we need a formula relating λ^τ to other invariants. It is easy to see that the homology lens space is given by (n/q) surgery along a knot k in an integral homology sphere Y, where q is relatively prime to n. Let Y_n denote the n-fold cyclic cover of Y branched along k. It is a homology sphere, and Σ is given by $1/q$ surgery on Y_n along a lift k_n of k.

Proposition 4. *The equivariant Casson invariant is given by the formula*

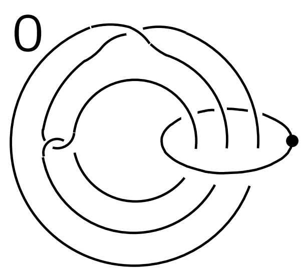
$$\lambda^\tau(\Sigma) = n\,\lambda(Y) + \frac{1}{8}\sum_{m=0}^{n-1} \sigma_{m/n}(k) + \frac{q}{2}\,\Delta_k''(1)$$
$$= \lambda^\tau(Y_n) + \frac{q}{2}\,\Delta_k''(1)$$

As above, $\lambda^\tau(Y_n)$ is the same as $\rho(Y_n)$, and moreover Σ is obtained by surgery on the lift of k to Y_n. So the conjectured equality $\lambda_{\mathrm{FO}}(S^1 \times_\tau \Sigma) \equiv \rho(\Sigma)$ can be deduced from the following lemma.

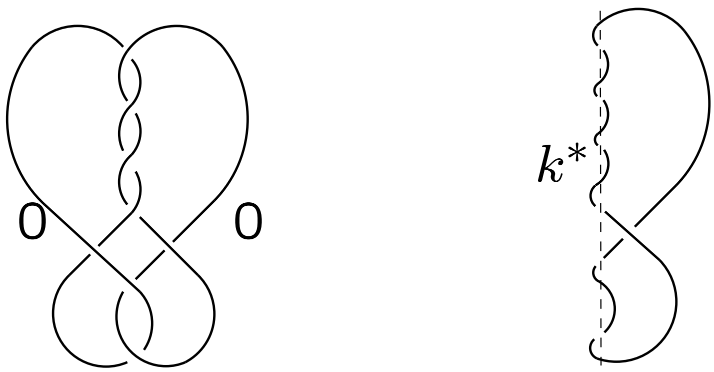
Lemma 5. *Let Y be in integral homology sphere and $\pi : Y_n \to Y$ its n–fold cyclic branched cover with branch set a knot k. Let k_n be the knot $\pi^{-1}(k)$ in Y_n. If Y_n is an integral homology sphere then* $\mathrm{arf}(k_n) = \mathrm{arf}(k)$ (mod 2).

Lefschetz numbers.

Theorem 3 computes (if $\mathcal{R}^*(\Sigma)$ is smooth) λ_{FO} in terms of the Lefschetz number of the action of τ on Floer homology. But this action is rather subtle! By definition, τ_* is computed by counting ASD connections on the mapping cylinder of τ. But you have to count these with signs, according to the orientation of the moduli space of ASD connections. So even if τ acts by the identity on $\mathcal{R}^*(\Sigma)$, it may be that the action on $HF_*(\Sigma)$ is not the identity.
Example ('Akbulut cork'):



The involution $\tau : \Sigma \to \Sigma$ simply interchanges the two link components, k_1 and k_2; this is best seen when the link is drawn in a symmetric form as below.



The representation variety is smooth, and the Floer homology of Σ is trivial in even degrees, and is a copy of $\mathbf{Z}$ in each of the odd degrees. Therefore, $\tau_* : HF_n(\Sigma) \to HF_n(\Sigma)$ is necessarily plus or minus the identity for each n.

The manifold Σ/τ is obtained from S^3 by surgery on the knot k^* which is the image of the link $k_1 \cup k_2$. The dotted line represents the branch set k; one can calculate its signature to be 16. From the formula given above, it follows that $\lambda^\tau(\Sigma) = 2$. Therefore the Lefschetz number of τ_* equals 4, and so τ_* itself must be $-\,$id in all degrees.

II. An approach to the conjecture $\lambda_{\mathbf{FO}} = \rho$.

Revisionist view of Casson's proof of $\lambda(\Sigma) = \rho(\Sigma)$: First, define versions of λ for manifolds M with $H_1(M) = \mathbf{Z}$, $\mathbf{Z}^2$, $\mathbf{Z}^3$. (Could go further, but other invariants are 0.) These are defined by counting flat SO(3) connections, or equivalently, projectively flat U(2) connections. Next, prove a surgery formula for a knot $K \subset M$, of the shape

$$\lambda(M(K, \pm 1)) = \lambda(M) \pm \lambda(M(K, 0))$$

Here $M(K, r)$ denotes the result of r-surgery on M along K. Observe that a similar formula holds for the Rohlin invariants, with care taken about which spin structures to use. Finally, prove directly that $\rho = \lambda$ for a 3-manifold with $H_1 = \mathbf{Z}^3$. We recently carried out this last step (cf. http://arXiv.org/abs/math/0302131), by a direct gauge-theoretic counting argument, to be described below.

We would like to carry out a similar program in dimension 4. The idea would be to work in a category of 4-manifolds X over S^1, i.e. with a preferred cohomology class $\alpha \in H^1(X;\mathbf{Z})$. What kind of surgeries should be allowed?

Definition: *(Asimov 1975). A round handle of index k is a pair*

$$S^1 \times (D^k \times D^{n-k-1}, S^{k-1} \times D^{n-k-1})$$

attached to a manifold with boundary.

There are obvious notions of round handle decompositions, round surgeries . . .

Asimov showed that the existence (in dimensions at least 4) of round handle decompositions, etc., is governed by the Euler characteristic. From this it is not hard to prove the following.

Proposition 6. *Let X be a smooth homology $S^1 \times S^3$. Then X may be obtained from $S^1 \times S^3$ via a series of round surgeries of index 2 and 3.*

Unfortunately, the intermediate stages in the resulting cobordism might not have well-defined Rohlin invariants (and their gauge theory invariants aren't so good either.) In order to make progress, we need to restrict the kind of surgeries we do.

Question: In the preceeding proposition, can we assume that the S^1 factor maps into X with degree ± 1? (The degree is by definition $\langle \alpha, S^1 \rangle$.)

Good news: It seems that there are surgery formulas for both λ and ρ invariants for round surgeries of degree ± 1.

Bad news: The question seems very hard, and probably the answer to the question as posed is negative.

III. Homology 4-tori.

By analogy with approach to Casson's invariant outlined above, the last step in the program should be to understand manifolds with the integer homology of of the 4-torus. So let X be a spin manifold with the same integer homology as T^4.

There's already a good candidate for the λ-type invariant: the Donaldson invariant D^w of degree 0. This counts ASD connections on an SO(3) bundle $P \to X$ with $w_2(P) \cup w_2(P) \equiv 0 \pmod 4$. The Rohlin invariant is a bit complicated to define; it depends *a priori* on the choice of $\alpha \in H^1(X)$. (Remember we are keeping track of such a class.) The class α determines an infinite cyclic cover $\tilde{X}_\alpha \to X$.

We need the following simple invariant of a homology n-torus Y: Choose a basis $\alpha_1, \ldots, \alpha_n$ for $H^1(Y; \mathbf{Z})$, and define $\det(Y) = |\langle \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n, [Y] \rangle|$.

The Rohlin invariant $\rho(X,\alpha)$ of a (spin) homology 4-torus (X,α) is defined if either

- det(X) is odd
- det(X) is even and $\tilde{X}_\alpha$ has finitely generated rational homology.

To define $\rho(X,\alpha)$, we recall that the $\mathbf{Z}_2$ cohomology of a manifold acts on its spin structures. Denote this action (for s a spin structure and $z \in H^1(Y;\mathbf{Z}_2)$ by $s \to s+z$. Choose any spin structure s on X, and an oriented 3-manifold Poincaré dual to α, and define

$$\rho(X,\alpha,s) = \rho(M, s_M)$$

This depends on the choice of spin structure on X. To eliminate this dependence, choose a basis $\{\alpha, x_1, x_2, x_3\}$ for $H^1(X)$, and define

$$\rho(X,\alpha) = \sum_{x \in \text{Span}\{x_1,x_2,x_3\}} \rho(X,\alpha,s+x)$$

Under either of the hypotheses above, $\rho(X,\alpha) \in \mathbf{Q}/2\mathbf{Z}$ is independent of the choice of M.

Theorem 7. *If the covering space $\tilde{X}_\alpha$ has the integral homology of T^3, then $\rho(X,\alpha) \in \mathbf{Z}/2\mathbf{Z}$, and $\rho(X,\alpha) \equiv \det(X) \pmod 2$.*

On the gauge-theoretic side, for w satisfying $w \cup w \equiv 0 \pmod 4$, define

$$\lambda(X,w) = \frac{1}{4} D^w(X).$$

With some details remaining to check, we have

Theorem 8. $\lambda(X,w) \equiv \det(X) \pmod 2$

Thus, modulo details, we have that $\rho(X,\alpha) \equiv \lambda(X,w) \pmod 2$. The idea of our program would is to 'propagate' this result back, via round surgeries, to relate ρ and λ_{FO} for a homology $\mathcal{S}^1 \times S^3$.

Finally, remark that Ruberman-Strle (2000) gave a result, similar to Theorem 8, for the Seiberg-Witten invariant.

Homology 3-tori.

The underlying ideas behind the theorem about 4-tori also work in dimension 3. Let Y be an integral homology T^3, with spin structure s. In this setting, we have a Rohlin invariant

$$\rho(Y) = \sum_{x \in H^1(Y;\mathbf{Z}_2)} \rho(Y, x+s).$$

Also, for any $w \in H^2(Y;\mathbf{Z}_2)$ we have a Casson-type invariant $\lambda(Y,w)$ defined by counting projectively flat U(2) connections on a bundle $P_c \to Y$ with $c_1(P_c) = c \equiv w \pmod 2$.

Theorem 9. *For any non-trivial* $w \in H^2(Y;\mathbf{Z}_2)$*,*

$$\lambda(Y,w) \equiv \det(Y) \equiv \rho(Y) \pmod 2$$

An integral version of the first equality, with a different definition of λ, is implicit in Casson's work, and was made explicit by C. Lescop (1992). The second congruence is essentially due to S. Kaplan (1979).

Our proof of the first congruence is a counting argument. Let $\mathcal{P}R^c(Y)$ denote the moduli space of projectively flat connections on the bundle P_c. There is an action (not free!) of $H^1(Y;\mathbf{Z}_2) = \mathbf{Z}_2^3$ on $\mathcal{P}R^c(Y)$, with quotient the moduli space of flat SO(3) connections.

We show the existence of enough perturbations to make $\mathcal{P}R^c(Y)$ smooth, that are equivariant with respect to this $H^1(Y;\mathbf{Z}_2)$ action. (This is potentially of some interest elsewhere in Floer theory; cf. comments in Donaldson's book.) The congruence is proved by looking at different kinds of orbits of the action. The relation with det(Y) comes from the following two facts:

- There are no connections fixed by all of $H^1(Y;\mathbf{Z}_2)$.
- There is a one connection fixed by a subgroup of order 4, if and only if det(Y) is odd.