Topologically slice knots with non-trivial Alexander Polynomial

Matt Hedden
Chuck Livingston
Daniel Ruberman

University of Miami
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A knot $K \subset S^3$ is trivial if it bounds an embedded disk in $S^3$.

**Problem:** Does every knot $K \subset S^3$ bound an embedded disk $D$ in $B^4$? Need extra structure (or else problem is trivial).
Require either
- $D$ is smooth:

or
- $D$ is flat: has a neighborhood homeomorphic to $D \times \mathbb{R}^2$.

We say that $K$ is smoothly (resp. topologically) slice.
Knots $K_i$, $i = 0, 1$ are *concordant* (smoothly or topologically) if there is an embedded $S^1 \times I \subset S^3 \times I$ with $K_i = S^1 \times \{i\}$.

Write $K_0 \sim_c K_1$ for smooth concordance; $K_0 \sim_c K_1 \iff K_1 \# - K_0$ slice.

- *Smooth* concordance group $C$: knots modulo smooth concordance.
- *Topological* concordance group $C_{\text{top}}$: knots modulo topological concordance.
Algebraic invariants.

Algebraic invariants of knots: Alexander polynomial $\Delta_K(t)$.

- **Fox-Milnor (1958):** $K$ slice $\Rightarrow \Delta_K = f(t)f(t^{-1})$.
  - Trefoil knot ($\Delta(t) = t^{-1} - 1 + t$) is not slice.

- Other algebraic invariants give surjection
  \[ A : C_{\text{top}} \rightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \]

- **Levine (1969):** In high dimensions, $A$ is an isomorphism.
- **Casson-Gordon (1975):** $A$ is not an injection.
  - **Jiang (1979):** $\mathcal{A} = \ker A$ is infinitely generated.
- **Cochran-Orr-Teichner (2003):** Deeper structure of $\mathcal{A}$.
  - Strengthened by Cha, Friedl, Harvey, Kim, Leidy.
Note that $\Delta_K(t) = 1$ implies $K \in \mathcal{A}$.

**Problem:** Is every knot with $\Delta_K(t) = 1$ slice?

Answer depends on smoothness:

**Theorem 1. Donaldson (1982):** There is no smooth 4-manifold with intersection form $E_8 \oplus E_8$.

Akbulut, Casson (and others) observed that Theorem 1 yields smoothly non-slice knots (pretzel knots) with $\Delta_K(t) = 1$.

One year later:

**Theorem 2. Freedman (1983):** $(\Delta_K(t) = 1) \Rightarrow K$ is topologically slice.

**Consequence:** $\mathcal{K} = \ker [\mathcal{C} \to \mathcal{C}_{\text{top}}]$ is nontrivial.
**Question:** What is the structure of $\mathcal{K}$? (Smooth concordance classes of topologically slice knots).

**Endo (1995):** $\mathbb{Z}^\infty \subset \mathcal{K}$, so it is infinitely generated.

All of these examples in $\mathcal{K}$ have trivial Alexander polynomial.

Easy to find examples with non-trivial polynomial: Choose $K \in \mathcal{K}$ and any knot $J$. Then $K \# J \# - J \in \mathcal{K}$, but has polynomial $\Delta_J(t)\Delta_J(t^{-1})$.

No new phenomenon here: $K \# J \# - J$ is smoothly concordant to $K$, and $\Delta_K = 1$!
Let $C_\Delta \subset K$ be the subgroup generated by knots with trivial Alexander polynomial. Is $C_\Delta = K$? In other words, are topologically slice, non-smoothly slice knots ‘explained’ by Freedman’s theorem?

**Theorem** (Hedden-Livingston-Ruberman): $K/C_\Delta$ is infinitely generated, and contains a $\mathbb{Z}^\infty$ subgroup.

The proof uses invariants from Heegaard-Floer theory.
Outline of proof.

Three Main Steps

1. **Find an Obstruction:** Theorem: If $K \sim_c J$, $\Delta_J(t) = 1$, then $d(M(K), s)$ constant for some set of $\text{Spin}^c$-structures $\{s\}$ on 2–fold branched cover of $K$, where $d$ is Heegaard-Floer correction term.

2. **Build Examples:** Construct a knot such that (1) $K$ is topologically slice, (2) $M(K)$ is simple enough to compute $d$.

3. **Compute the Obstruction:** Use Heegaard-Floer knot homology to compute $d(M(K), s)$. Extend to multiples and connected sums of examples by understanding $\text{Spin}^c$ structures on $M(\#_i n_i K_i)$. 
Step A: 2–fold cyclic branched covers of slice knots: \( M(K) \).

- \( H_1(M(K), \mathbb{Z}_2) = 0 \).

- If \( K \) is slice, \( M(K) = \partial W^4 \) for some \( \mathbb{Z}_2 \)–homology ball: \( H_\ast(W, \mathbb{Z}_2) = 0 \).

- In this case, \( |H_1(M(K))| = |\text{Ker}|^2 \) where

  \[
  \text{Ker} = \text{Ker}( H_1(M(K)) \to H_1(W) ).
  \]

- \( \text{Spin}^c(M(K)) \leftrightarrow H_1(M(K)) \).

- \( \text{Ker} \) corresponds to \( \text{Spin}^c \) structures that extend to \( W \).
Obstructions: Branched covers and $d(M(K), s)$.

Step B: Implications of $K$ concordant to $J$, with $\Delta_J(t) = 1$.

- If $K \sim_c J$, then $K\# - J$ is slice.

- $\Delta_J(t) = 1 \Rightarrow H_1(M(J)) = 0$.

- $M(K\# - J) = M(K)\#\Sigma^3$ bounds $\mathbb{Z}_2$-homology 4–ball $W$, where $\Sigma^3$ is a $\mathbb{Z}$–homology 3–sphere.

- Thus, the problem is reduced to showing that particular $M^3$ have property that $M^3\#\Sigma^3$ does not bound a $\mathbb{Z}_2$–homology ball for any $\mathbb{Z}$–homology sphere $\Sigma^3$. 


Step C: Use the Heegaard-Floer “correction term” of Ozsváth-Szabó, $d(N^3, s)$, where $s$ is a Spin$^c$–structure.

**Theorem:** If $M(K)\#\Sigma^3$ bounds a $\mathbb{Z}_2$–homology ball $W$ for some $\Sigma^3$, then $d(M(K), s)$ is constant on a nontrivial subgroup (square root order) of $H_1(M(K))$.

- If $(N^3, s)$ bounds $\mathbb{Z}_2$–homology ball $(Y^4, s')$, then $d(N^3, s) = 0$.
- $d(M(K)\#\Sigma^3, s) = d(M(K), s) + d(\Sigma^3, 0) = 0$ for all $s$ in $\text{Ker} = \text{Ker} \left( H_1(M(K)) \to H_1(W) \right)$.
- $d(M(K), s)$ is constant for Spin$^c$–structures $s$ corresponding to elements in $\text{Ker}(H_1(M(K)) \to H_1(W))$. 
The basic example: $K_J$.

If $J$ is unknotted then $K_J$ is slice (surger the evident punctured Klein bottle).

If $J$ is slice (topologically) then $K_J$ is slice (topologically).

We will let $J$ be a $\Delta = 1$ knot, known not to be smoothly slice. (A connected sum of untwisted doubles of the trefoil: $D(T(2, 3))$.)
Whitehead double of $T(2, 3)$. 
Basic example: \( K_J \) and its 2–fold cover \( M(K_J) \).

Akbulut-Kirby gave algorithm for describing 2–fold cover as surgery on a link.
**Basic example:** $K_J$ and its 2–fold cover $M(K_J)$.

$M(K_J) = 9$–surgery on $T(2, 3) \# J \# J$.

$H_1(M(K_J)) = \mathbb{Z}_9$. If $M(K_J) \# \Sigma^3$ bounds $\mathbb{Z}_2$–homology ball $W$, $\text{Ker } (H_1(M) \to H_1(W)) = \mathbb{Z}_3 \subset \mathbb{Z}_9$. 
To any 3–manifold $M$ and Spin$^c$–structure $s$ there is a graded chain complex $CF^\infty(M, s)$. This is a $\mathbb{Z}_2[U, U^{-1}]$–module, $U$ lowers grading by 2.

Divide by subcomplex $CF^-(M, s)$ gives quotient $CF^+(M, s)$, with homology $HF^+(M, s)$. This is a $\mathbb{Z}_2[U]$–module, $U$ lowers grading by 2.

$d(M, s)$ is defined to be least grading of a nontrivial element in $HF^+(M, s)$ (in image of $U^k$ for large $k$).

Picture on next slide shows $HF^+(S^3, s_0) = \mathbb{Z}_2[U]$, generated by an element $a$ in grading 0. So $d(S^3, s_0) = 0$. 
**Example:** $CF^\infty(S^3)$.

\[ CF^\infty(S^3) = \mathbb{Z}_2[U, U^{-1}] \]

\[ CF^+(S^3) = \mathbb{Z}_2[U] \]

Note: boundary operators in chain complex trivial, so $HF = CF$. 

Computing $HF^+(S_q^3(K))$.

- If $K \subset S^3$ is a knot, then $K$ yields a filtration on a chain complex for $HF(S^3)$, giving a doubly filtered chain complex $CFK^\infty(S^3, K)$.
- For $q$ large (compared to genus of $K$), $HF^+(S_q(K), s)$ is determined by $CFK^\infty$. ($S_q(K)$ is $q$–surgery on $K$.)
- We are interested in $T(2, 3)\#nD(T(2, 3))$. Ozsváth-Szabó determined $CFK^\infty(S^3, T(p, q))$ and Hedden determined (much of) $CFK^\infty(S^3, D(T(2, 3)))$. 
**Example:** $\text{CFK}^\infty(S^3, T(4,5))$.

$$HF^\infty(S^3) = \mathbb{Z}_2[U, U^{-1}].$$
**Example:** $CFK^+(S^3, T(4, 5))$.

\[HF^+(S^3) = \mathbb{Z}_2[U, U^{-1}]/U^{-1}\mathbb{Z}_2[U]. \ a_{0,6} \text{ has least grading.}\]
**Example:** \( HF^+(S_q(T(4, 5), s_3)) \).

\[
HF^+(S_q(T(4, 5), s_3)) = \mathbb{Z}_2[U, U^{-1}]/U^{-1}\mathbb{Z}_2[U]. \quad c_{0, 2} \text{ has least grading.}
\]
Basic example: computation of $d(M(K), s_i)$. 

In the first case we are interested in, 9–surgery on $L_3 = T(2, 3)\#4D(T(2, 3))$, the chain complex looks something like 

$$CFK^\infty(S^3, T(2, 3)) \otimes (CFK^\infty(S^3, T(2, 3)) \oplus \text{acyclic}) \otimes 4.$$ 

An analysis of gradings is sufficient to show $d(S^3_9(K_3, s_0)) = -4$ and $d(S^3_9(K_3, s_3)) = -2$. 
Diagram.

\[ C(4,5) \otimes C(2,3)^\otimes 7 \]
Infinite families of examples.

More generally, to get infinite linearly independent examples, we are interested in, $p^2$–surgery on $L_p = T(p - 1, p)\# \frac{p - 1}{2} D(T(2, 3))$, chain complex looks something like

$$CFK^\infty(S^3, T(p - 1, p)) \otimes (CFK^\infty(S^3, T(2, 3)) \oplus \text{acyclic}) \otimes \frac{3p - 1}{2}.$$

In this case, an analysis of gradings is sufficient to show $d(S^3_{p^2}(K_3, s_0)) \leq -p - 1$ and $d(S^3_{p^2}(K_3, s_p)) \geq -p + 1$.  

Friedl-Teichner (2005) constructed a family of topologically slice knots. Theorem (Friedl-Teichner, 2005): If $\Delta_K(t) = (2t - 1)(t - 2)$ then certain homological conditions imply $K$ is topologically slice.

$$\text{Ext}^1_{Z[G]}(H_1(S^3 - K; Z[G]), Z[G]) = 0, \quad G = Z \rtimes Z[1/2].$$

- Are the F-T examples smoothly slice?
- If so, are they smoothly concordant to $\Delta = 1$ knots?

The simplest examples are in fact concordant to knots with trivial Alexander polynomial; the very simplest of these are not smoothly slice.
Three dimensional $\mathbb{Z}_2$ bordism.

- Let $\Omega^3_{\mathbb{Z}_2}$ denote the group of $\mathbb{Z}_2$ homology 3–spheres modulo smooth, $\mathbb{Z}_2$ homology cobordisms.

- If $M$ is an integral homology 3–sphere, then $M$ bounds a contractible topological 4–manifold. (Freedman)

- The kernel $\text{Ker} = \text{Ker}(\Omega^3_{\mathbb{Z}_2} \rightarrow \Omega^3_{\text{Top}})$ is infinitely generated.

- The manifolds $M(K_p)$ constructed earlier provide examples to show that $\text{Ker} / \Omega^3_{\mathbb{Z}}$ is infinitely generated, where $\Omega^3_{\mathbb{Z}}$ is the subgroup generated by integral homology spheres.