Periodic-end Dirac operators and Seiberg-Witten theory*

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*Preliminary report

The simplest smooth 4-manifolds

- Simply connected: S^4, $\mathbb{CP}^2$, $S^2 \times S^2$.
- Non-simply connected: $S^1 \times S^3$.

Will concentrate on invariants of manifolds X with the homology of $S^1 \times S^3$. Classical $\mathbb{Z}_2$-valued invariant $\rho(X)$ arising from Rohlin's signature theorem.

- Choose oriented $M^3 \subset X$ generating $H_3(X)$.
- Choose spin 4-manifold W with $\partial W = M$
- $\rho(X) = \rho(M) = \frac{1}{8}\sigma(W)$

Long-term goal to find $\mathbb{Z}$-valued lift of $\rho(X)$.

- Applications to classification of manifolds.
- Applications to homology cobordism and triangulation of high-dimensional manifolds.

Approach is to calculate $\rho(X)$ analytically via gauge theory–Yang Mills and Seiberg-Witten theory.

Seiberg-Witten equations

Seiberg-Witten theory assigns to a 4-manifold Y and Spin^c structure $\mathfrak{s}$, a number $SW(Y, \mathfrak{s})$, by counting irreducible solutions (up to gauge equivalence) to the Seiberg-Witten equations.

Variables: Spin^c connection A, spinor $\psi \in C^\infty(S^+)$, and $r \in \mathbb{R}^+$

$$D_A^+(g)\psi = 0 \qquad \int_Y |\psi|^2 = 1$$

$$F_A^+ + r^2 q(\psi) = \mu$$

where g is a metric on Y, and $\mu \in \Omega_+^2(Y; i\mathbb{R})$.

A solution is irreducible if $r \neq 0$.

Seiberg-Witten equations

- Equations depend on metric on Y and 2-form μ.
- Generic perturbation μ makes moduli space smooth, oriented 0-manifold.
  - Version of equations with r yield ‘blown-up’ moduli space of Kronheimer-Mrowka.
- Count irreducible ($r \neq 0$) solutions to μ-perturbed Seiberg-Witten equations.
  - Independent from g and μ if $b_2^+ Y > 1$.

Specialize to case of X with homology of $S^1 \times S^3$, and write $\mu = d^+\beta$. The algebraic count of irreducible solutions is denoted $\mathrm{SW}(X, g, \beta)$.

$$\boxed{\text{Key problem: } \mathrm{SW}(X, g, \beta) \text{ depends on } g \text{ and } \beta.}$$

- Consider $\mathrm{SW}(X, g_t, \beta_t)$ for 1-parameter family (g_t, β_t).
- Since $b_2^+(X) = 0$, may have solutions (A_t, r_t, ψ_t) with $r_t \to 0$ as $t \to t_0$, so count can change.
- Want some other metric-dependent term with similar jump.
- For $X = S^1 \times M^3$, done by Chen (1997) and Lim (2000).
- Counter-term from η-invariants of Dirac operator and signature operator on M^3.

Proposed counter-term in non-product case:

$$\boxed{\textbf{Index of Periodic-end Dirac operator.}}$$

Setup: Closed spin manifold X with a map $f : X \to S^1$, surjective on π_1. This gives

- Connected $\mathbb{Z}$-cover $\tilde{X} \to X$, and lift $t : \tilde{X} \to \mathbf{R}$ of f.
- Dirac operator $\tilde{D}^+ : C^\infty(\tilde{S}^+) \to C^\infty(\tilde{S}^-)$.
- For any regular value $\theta \in S^1$ for f, a submanifold $f^{-1}\theta = M \subset X$.

Question: *When is $\tilde{D}^+$ a Fredholm operator?*

To make sense of this, need to complete $C_0^\infty(\tilde{S}^\pm)$ in some norm. Pick $\delta \in \mathbf{R}$, and define

$$L^2_\delta(\tilde{S}^\pm) = \{s \mid \int_{\tilde{X}} e^{t\delta}|s|^2 < \infty\}$$

as well as Sobolev spaces $L^2_{k,\delta}(\tilde{S}^\pm)$.
Should really ask if the dimensions of the kernel/cokernel of

$$\tilde{D}^+ : L^2_{k,\delta}(\tilde{S}^\pm) \to L^2_{k-1,\delta}(\tilde{S}^\pm)$$

are finite. If so we'll be sloppy and say $\tilde{D}^+$ is Fredholm on L^2_δ.
The most useful case for us is $\delta = 0$.

Taubes' idea: Fourier-Laplace transform

$$s \Rightarrow \hat{s}_\mu e^{\mu t(x)} \sum_{n=-\infty}^{\infty} e^{\mu n} s(x+n) \text{ for } \mu \in \mathbb{C}$$

converts to family of problems on *compact* X. For each $c \in \mathbb{C}$, have the *twisted* Dirac operator $D_c^+ : C^\infty(S^+) \to C^\infty(S^-)$ given by

$$D_c^+ s = D^+ s - \log(c)\, dt \cdot s.$$

Theorem 1 (Taubes, 1987)

Fix $\delta \in \mathbf{R}$*. Suppose that* $\ker D_c^+ = \{0\}$ *for all* $c \in \mathbb{C}^*$ *with* $|c| = e^{\frac{\delta}{2}}$*. Then* $\tilde{D}^+$ *is Fredholm on* L^2_δ*.*

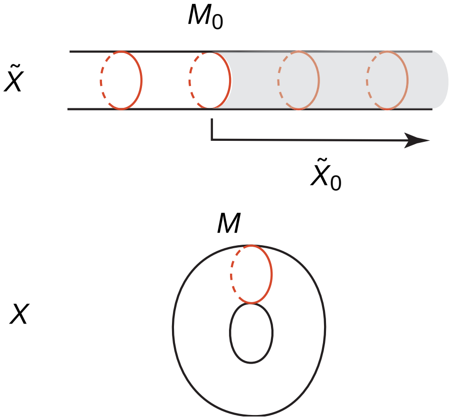
Theorem 2 (R-Saveliev, 2006)

For a generic metric on X, the operator $\tilde{D}^+$ is Fredholm on L^2.

- Suffices to find one metric with D_c invertible $\forall c \in S^1$.
- We apply technique of Ammann-Dahl-Humbert (2006).
    - Invertibility of D_c, $\forall c \in S^1$, can be pushed across a cobordism.

End-periodic manifolds

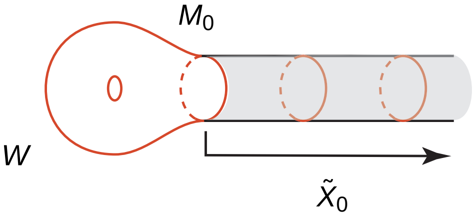
End-periodic manifolds are periodic in finitely many directions, each modeled on a $\mathbb{Z}$ covering $\tilde{X} \to X$. Let $M \subset X$ be non-separating; it lifts to a compact submanifold $M_0 \subset \tilde{X}$.



End-periodic manifolds

Let $\tilde{X}_0$ be everything to the right of M_0, and choose a compact oriented spin manifold W with (oriented) boundary $-M$. From these pieces, form the *end-periodic manifold* with end modeled on $\tilde{X}$:

$$Z = W \cup_{M_0} \tilde{X}_0$$



Excision principle: Everything we said about Dirac operators on $\tilde{X}$ holds for Dirac operators on Z.

- For metric g on X, extending to metric on Z, get Dirac operator $D^+(Z, g)$ and twisted version $D^+_\beta(Z, g)$ for $\beta \in \Omega^1(X; i\mathbb{R})$.
- Fredholm on L^2 for generic (g, β).
- $\mathrm{ind}(D^+_\beta(Z, g))$ depends on choice of W in simple way.
- *Unlike compact case,* $\mathrm{ind}(D^+_\beta(Z, g))$ *depends on* (g, β)*.*
    - Can jump in family g_t if $\ker(D^+_c(X, g_0)) \neq \{0\}$ for $c \in S^1$.

Observation: $\operatorname{ind}(D^+_\beta(Z,g))$ jumps at the same place as $SW(X,g,\beta)$. This suggests that we try to use one to balance the other. Have to get rid of dependence of $\operatorname{ind}(D^+_\beta(Z,g))$ on compact manifold W.

Provisional definition: Consider the quantity

$$\lambda_{SW}(X,g,\beta) = SW(X,g,\beta) - \operatorname{ind}(D^+_\beta(Z,g)) - \frac{1}{8}\operatorname{sign}(W)$$

Remark: Previous work (R-Saveliev 2004) defines $\lambda_{Don}(X)$ by counting flat connections.

Conjecture 3

$\lambda_{SW}(X,g)$ is metric-independent and equals $\lambda_{Don}(X)$.

Will discuss approach to independence part of Conjecture 3 shortly.

Properties of λ_{SW}

1. Independence from various choices
    - Choice of slice $M \subset X$ and lift $M_0 \subset \tilde{X}$.
    - Choice of W with $\partial W = M$, and extension of metric over W.
2. Reduction mod 2 of λ_{SW} is classical Rohlin invariant $\rho(X)$.

Item 1: excision principle.
Item 2: two ingredients. Involution in Seiberg-Witten theory makes $SW(X, g)$ even, and quaternionic nature of Dirac operator makes $\mathrm{ind}(D^+(Z, g))$ even.

Have seen that in a family g_t, the invariants $SW(X, g_t, \beta_t)$ and $\operatorname{ind}(D^+_{\beta_t}(Z, g_t))$ jump at the same t. Change in $SW(X, g, \beta)$ understood: wall-crossing phenomenon in gauge theory.

If $X = S^1 \times M^3$, then change in index is ‘spectral flow’ of Dirac operators on M, studied by Atiyah-Patodi-Singer. Conjecture 3 proved in this situation independently by Chen and Lim.

General periodic case more subtle; there’s no operator on M or spectrum to flow.

What we know so far: Somewhat easier, but basically equivalent to fix metric g, and vary the exponential weight. Consider fixed operators D^+ on L^2_δ as δ runs over the interval $[\delta_0, \delta_1]$. When Fredholm, denote its index by $\mathrm{ind}_\delta(D^+)$.

Denote by $\mathcal{S}(\delta_0, \delta_1)$ the set of $z \in \mathbb{C}$ with $\ker(D_z) \neq 0$ and $e^{\delta_0/2} < |z| < e^{\delta_1/2}$. By Taubes' theorem 1, this is a finite set. To each $z \in \mathcal{S}(\delta_0, \delta_1)$, we associate a 'multiplicity' $d(z)$. Definition of $d(z)$ complicated; count of solutions to some system of equations. But we can show

Lemma 4

If $\dim \ker(D_z^+) = 1$, *then* $d(z) = 1$.

Theorem 5

For generic metric g, the difference

$$\operatorname{ind}_{\delta_1}(D^+(Z,g)) - \operatorname{ind}_{\delta_0}(D^+(Z,g)) = \sum_{z \in \mathcal{C}(\delta_0,\delta_1)} d(z)$$

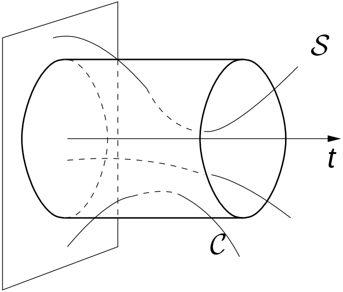
So what's left to do? Geometric case–fix $\delta = 0$, vary (g, β).

- Translate back to fixed (g, β) and varying weights δ.
- Change in δ from local description of SW moduli space.
- Relate $d(z)$ to wall-crossing signs in SW theory.

Pictorial interpretation:

Fix $\delta = 0$, let (g_t, β_t) vary. Write $\mathcal{C} = S^1 \times [0,1]$; this is where changes in SW and $\operatorname{ind}(D^+_\beta(Z, g))$ occur.

Let $\mathcal{S} = \{(c,t) \in \mathbb{C} \times [0,1] \mid \ker(D^+_{\beta_t}(X, g_t) - \log(c)\, dt) \neq 0\}$



Then we basically want to show $\Delta SW = \mathcal{S} \cdot \mathcal{C}$.

Positive scalar curvature

Basic differential geometry: (X, g) Riemannian manifold

$\Longrightarrow$ Riemannian curvature tensor

$\xrightarrow{\text{tr}}$ scalar curvature S_g.

Question: *Which manifolds have a metric g with $S_g > 0$?*

We say that g is a metric of positive scalar curvature (PSC).
Not all manifolds admit metrics with PSC:

- Dirac operators (Lichnerowicz; Gromov-Lawson)
- Minimal surfaces (Schoen-Yau) in all dimensions
- Gauge theory (Seiberg-Witten) in dimensions 3 and 4.

Some non-orientable 4-manifolds.

Let Y be the non-orientable S^3 bundle over S^1, a.k.a. $S^1 \times_\rho S^3$ where ρ is a reflection.

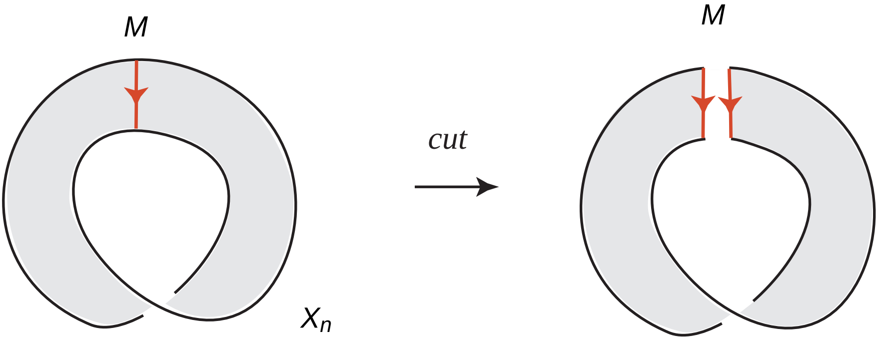
Question: *Is there a smooth manifold Y' homotopy equivalent to Y, but not diffeomorphic to Y?*

This is still unknown, but for $k > 0$, we can consider instead $Y_k = Y \#_k S^2 \times S^2$. There are manifolds $Y'_k \simeq Y_k$ with $Y'_k \not\cong Y_k$ constructed by Cappell-Shaneson, Akbulut, and Fintushel-Stern. The difference between Y'_k and Y_k stems from Rohlin's theorem.

Note that all of the Y_k admit a metric of PSC. We will use end-periodic Dirac operators to show that the exotic Y_k' do not admit PSC metrics.

Let's assume that we have a non-orientable manifold X_n with a map $f: X_n \to S^1$ such that $w_2(X_n) = 0$ and $w_1(X_n)$ is the pull-back of the generator of $H^1(S^1)$. As before, we get a submanifold $M = f^{-1}\theta$, and we can cut along M as before to get the *orientable* manifold $V = X_n - \text{nhd}(M)$.

Choose an orientation of V, then $\partial V = 2$ copies of M as shown below. It's not hard to show that in fact V has a spin structure, and so its boundary acquires one as well.



Following Cappell-Shaneson, define

$$\alpha(X_n) = \rho(M) - \frac{1}{16}\sigma(V) \in \mathbf{Q}/2\mathbb{Z}$$

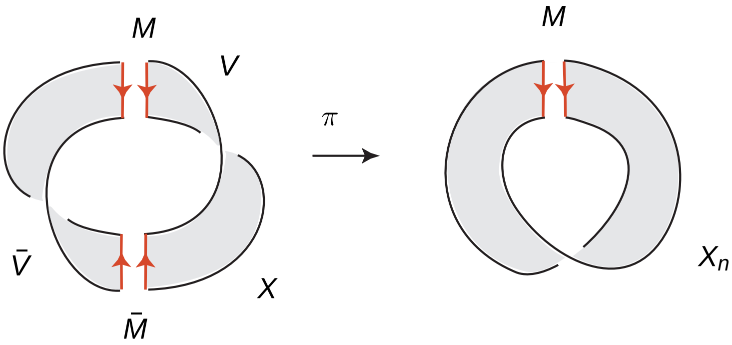
which does not depend (up to sign) on choices made. For manifolds homotopy equivalent to Y_k, it turns out that $\alpha \equiv 0 \text{ or } 1 \pmod{2\mathbb{Z}}$. Cappell-Shaneson used a similar invariant to detect their exotic $\mathbf{RP}^4$.

Theorem 6

(R-Saveliev, 2006) Suppose that $\alpha(X_n) \neq 0$. Then X_n admits no metric of positive scalar curvature.

Proof: Suppose that X_n does admit a PSC metric g_n. The idea is to use this to build a periodic-end manifold with positive scalar curvature on its end, and to use properties of the index of the Dirac operator to show that α must vanish. We continue with notation from before: M is a codimension-one submanifold of X_n, and V is X_n cut along M, with an orientation chosen. First, consider the orientation double cover $\pi : X \to X_n$; note that X is canonically oriented. Since X is locally the same as X_n, the metric $g = \pi^* g_n$ has PSC. There are two lifts of V to X, but we can single one out by requiring that π preserve the orientation.

So we get the following picture



Now, choose a spin manifold W^4 with boundary M, and consider the periodic-end manifold (modeled on $\tilde{X} \to X$):

$$W \cup_M (\bar{V} \cup_{\bar{M}} V) \cup_M (\bar{V} \cup_{\bar{M}} V) \cup_M \cdots$$

Since g has PSC, the index of the Dirac operator on this manifold makes sense, and we define

$$\alpha_{\text{Dirac}} = \operatorname{ind} D(W \cup (\bar{V} \cup V)) \cup \cdots) + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V)$$

This is not much of an invariant: it might depend on the choice of g_n, and on the choice of M (and hence V). But, excision implies that α_{Dirac} does <u>not</u> depend on W.

Using this independence, we calculate

$$\begin{aligned}\alpha_{\text{Dirac}} &= \operatorname{ind} D(W \cup (\bar{V} \cup V)) \cup \cdots) \\ &\quad + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V) \\ &= \operatorname{ind} D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots) \\ &\quad + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V) \\ &= \operatorname{ind} D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots) \\ &\quad + \frac{1}{8}\sigma(W \cup \bar{V}) + \frac{1}{16}\sigma(V)\end{aligned}$$

where in the last line we used that $\sigma(W \cup \bar{V}) = \sigma(W) - \sigma(V)$.

Using excision, replace $W \cup \bar{V}$ by $\bar{W}$ to get

$$\begin{aligned}\alpha_{\text{Dirac}} &= \operatorname{ind} D(\bar{W} \cup (V \cup \bar{V}) \cup \cdots) \\ &\quad + \frac{1}{8}\sigma(\bar{W}) + \frac{1}{16}\sigma(V) = -\alpha_{\text{Dirac}}\end{aligned}$$

and we conclude that $\alpha_{\text{Dirac}} = 0$!

Finally, recall that the quaternionic nature of the Dirac operator implies (even on non-compact manifolds) that its index is even. So the mod 2 reduction of

$$\alpha_{\text{Dirac}} = \operatorname{ind}(D) + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V)$$

is the Cappell-Shaneson invariant α, which must then vanish as well (mod 2). ☐