Periodic-end Dirac Operators and Positive Scalar Curvature

Daniel Ruberman
Nikolai Saveliev
Recall from basic differential geometry: \((X, g)\) Riemannian manifold \(\Rightarrow\) Riemannian curvature tensor \(\text{tr}\) \(\Rightarrow\) scalar curvature \(S_g\).

**Question:** Which manifolds have a metric \(g\) with \(S_g > 0\)?

We say that \(g\) is a metric of positive scalar curvature (PSC). Not all manifolds admit metrics with PSC, as can be shown by many techniques: Dirac operators (Lichnerowicz; Gromov-Lawson) and minimal surfaces (Schoen-Yau) in all dimensions; gauge theory (Seiberg-Witten) special to dimensions 3 and 4.

Will describe a new technique to address this problem for some 4-manifolds, based on the analysis of the Dirac operator on non-compact manifolds.
I. Brief review of Dirac operators

Let $X^n$ be an oriented Riemannian manifold with a spin structure. Recall that this means that $w_2(X) = 0$ and that there is a complex vector bundle $S \to X$, and a ‘Clifford multiplication’ $T^*X \otimes S \to S$. Some features of this:

• Clifford relation: if $\alpha_1, \alpha_2 \in T^*X$, and $s \in S$, $\alpha_1 \cdot (\alpha_2 \cdot s) + \alpha_2 \cdot (\alpha_1 \cdot s) = -2 \langle \alpha_1, \alpha_2 \rangle s$.

• If $n = \dim X$ is even then $S \cong S^+ \oplus S^-$, with $T^*X \otimes S^\pm \to S^\mp$.

• If $n$ is divisible by 4 then $S^\pm$ are quaternionic vector spaces, and Clifford multiplication is quaternionic linear.

• The metric on $X$ gives a connection on $S$: for every tangent vector $w$ on $X$, a derivation $\nabla_w : C^\infty(S) \to C^\infty(S)$.
Dirac operator \( D : C^\infty(S) \to C^\infty(S) \):

\[
Ds = \sum_i e_i \cdot \nabla e_i s, \quad \{e_i\} \text{ an ON basis for } T^*_X
\]

Key facts when \( X \) is closed:

- \( \dim \ker D, \ \dim \ker D^* < \infty \). In other words, \( D \) is a Fredholm operator.

- If \( n = \dim X \) is even, then \( D : C^\infty(S^\pm) \to C^\infty(S^{\mp}) \), ie \( D = D^+ \oplus D^- \). We define \( \text{ind } D = \dim_C \ker D^+ - \dim_C \ker D^- \).

- Lichnerowicz-Weitzenböck formula:

\[
D^* D = \nabla^* \nabla + \frac{1}{4} S_g
\]

- The operator \( \nabla^* \nabla \) is non-negative. Hence \( S_g > 0 \Rightarrow \text{ind } D = 0 \).

- Likewise for Dirac twisted by flat bundle.
II. Periodic manifolds and Dirac operators.

Consider a closed manifold $X$ with a map $f : X \to S^1$. This gives

- A $\mathbb{Z}$-cover $\tilde{X} \to X$, and lift $t : \tilde{X} \to \mathbb{R}$ of $f$.

- If $X$ is spin, $\tilde{D}^+ : C^\infty(\tilde{S}^+) \to C^\infty(\tilde{S}^-)$.

- For any regular value $\theta \in S^1$ for $f$, a submanifold $f^{-1}\theta = M \subset X$.

**Question:** Under what circumstances is $\tilde{D}^+$ a Fredholm operator?

To make sense of this, need to complete $C^\infty(\tilde{S}^\pm)$ in some norm. Pick $\delta \in \mathbb{R}$, and define

$$L^2_\delta(\tilde{S}^\pm) = \{ s | \int_{\tilde{X}} e^{t\delta} |s|^2 < \infty \}$$

Likewise, get Sobolev spaces $L^2_{k, \delta}(\tilde{S}^\pm)$.
Should really ask if the dimensions of the kernel/cokernel of $\tilde{D}^+ : L^2_{k,\delta}(\tilde{S}^{\pm}) \to L^2_{k-1,\delta}(\tilde{S}^{\pm})$ are finite. If so we’ll be sloppy and say $\tilde{D}^+$ is Fredholm on $L^2_{\delta}$. The most useful case (for us) is $\delta = 0$.

**Taubes’ idea:** Fourier-Laplace transform converts to family of problems on compact $X$. For each $c \in \mathbb{C}$, have the twisted Dirac operator $D_c : C^\infty(S) \to C^\infty(S)$ given by

$$D_c s = Ds + ic \, dt \cdot s.$$ 

**Theorem 1.** *(Taubes, 1987)* Fix $\delta \in \mathbb{R}$. Suppose that $\ker D_c = \{0\}$ for all $c \in \mathbb{C}^*$ with $|c| = e^{\frac{\delta}{2}}$. Then $\tilde{D}^+$ is Fredholm on $L^2_{\delta}$.

**Corollary 2.** If $X$ has a Riemannian metric of positive scalar curvature, then $\tilde{D}^+$ is Fredholm on $L^2_{\delta}$ for any $\delta$. 
This theorem, originally proved (more directly) by Gromov-Lawson (1983), is not per se an obstruction to existence of PSC metrics.

**Theorem 3.** (R-Saveliev, 2006)

*For a generic metric on $X$, the operator $\tilde{D}^+$ is Fredholm on $L^2$.*

There are many manifolds (eg $T^4$) which admit no PSC metric, to which theorem 3 applies.

III. End-periodic manifolds.

We are really interested in manifolds which are periodic in just one direction. Start with a $\mathbb{Z}$ covering $\tilde{X} \to X$, and consider a submanifold $M \subset X$ as pictured below. $M$ does not separate $X$, and lifts to a compact submanifold $M_0 \subset \tilde{X}$. 
Let $\tilde{X}_0$ be everything to the right of $M_0$, and choose a compact oriented spin manifold $W$ with (oriented) boundary $-M$. From these pieces, form the end-periodic manifold with end modeled on $\tilde{X}$:

$$Z = W \cup_{M_0} \tilde{X}_0$$

**Excision principle:** Everything we said about Dirac operators on $\tilde{X}$ holds for Dirac operators on $Z$. 
IV. Invariants of non-orientable 4-manifolds.

Let $Y$ be the non-orientable $S^3$ bundle over $S^1$, a.k.a. $S^1 \times_\rho S^3$ where $\rho$ is a reflection. More generally, let $Y_k = Y \#_k S^2 \times S^2$. Note that all of the $Y_k$ admit a metric of PSC.

**Question:** Is there a smooth manifold $Y'$ homotopy equivalent to $Y$, but not diffeomorphic to $Y$?

This is still unknown, but for $k > 0$, there are manifolds $Y'_k \simeq Y_k$ with $Y_k \not\cong Y_k$ constructed by Cappell-Shaneson, Akbulut, and Fintushel-Stern. We will use end-periodic Dirac operators to show that these exotic manifolds do not admit PSC metrics.

The difference between $Y'_k$ and $Y_k$ stems from Rohlin’s theorem.
Rohlin’s Theorem: Let $X$ be a smooth, closed, oriented spin 4-manifold. Then the signature $\sigma(X)$ is divisible by 16.

The fastest proof of this uses the index theorem (twice!) to show that the index of the Dirac operator is given by $\frac{\sigma(X)}{8}$. But as remarked earlier, for a 4-manifold that index is even, because the kernel and cokernel are quaternionic vector spaces.

Rohlin’s theorem gives rise to the Rohlin invariant of a spin 3-manifold $(M, s)$. Let $M = \partial W$ where the spin structure $s$ extends over $W$. Then we define

$$\rho(M, s) = \frac{\sigma(X)}{8} \in \mathbb{Q}/2\mathbb{Z}.$$  

Let’s assume that we have a non-orientable manifold $X_n$ with a map $f : X_n \to S^1$ such that $w_2(X_n) = 0$ and $w_1(X_n)$ is the pull-back of the generator of $H^1(S^1)$. As before, we get a submanifold $M = f^{-1}\theta$, and we can cut along $M$ as before to get the orientable manifold $V = X_n - \text{nhd}(M)$.  

Choose an orientation of $V$, then $\partial V = 2$ copies of $M$ as shown below. It’s not hard to show that in fact $V$ has a spin structure, and so its boundary acquires one as well.

Following Cappell-Shaneson, define

$$\alpha(X_n) = \rho(M) - \frac{1}{16}\sigma(V) \in \mathbb{Q}/2\mathbb{Z}$$

which does not depend (up to sign) on choices made. For manifolds homotopy equivalent to $Y_k$, it turns out that $\alpha \equiv 0$ or $1 \pmod{2\mathbb{Z}}$. Cappell-Shaneson used a similar invariant to detect their exotic $\mathbb{RP}^4$. 

11
V. A new obstruction to existence of PSC metrics.

**Theorem 4.** *(R-Saveliev, 2006)* Suppose that $\alpha(X_n) \neq 0$. Then $X_n$ admits no metric of positive scalar curvature.

**Proof:** Suppose that $X_n$ does admit a PSC metric $g_n$. The idea is to use this to build a periodic-end manifold with positive scalar curvature on its end, and to use properties of the index of the Dirac operator to show that $\alpha$ must vanish. We continue with notation from before: $M$ is a codimension-one submanifold of $X_n$, and $V$ is $X_n$ cut along $M$, with an orientation chosen.

First, consider the orientation double cover $\pi : X \to X_n$; note that $X$ is canonically oriented. Since $X$ is locally the same as $X_n$, the metric $g = \pi^* g_n$ has PSC. There are two lifts of $V$ to $X$, but we can single one out by requiring that $\pi$ preserve the orientation.
So we get the following picture

Now, choose a spin manifold $W^4$ with boundary $M$, and consider the periodic-end manifold (modeled on $\tilde{X} \to X$):

$$W \cup_M (\tilde{V} \cup_{\tilde{M}} V) \cup_M (\tilde{V} \cup_{\tilde{M}} V) \cup_M \cdots$$

Since $g$ has PSC, the index of the Dirac operator on this manifold makes sense, and we define

$$\alpha_{\text{Dirac}} = \text{ind } D(W \cup (\tilde{V} \cup V) \cup \cdots) + \frac{1}{8} \sigma(W) - \frac{1}{16} \sigma(V)$$

This is not much of an invariant: it might depend on the choice of $g_n$, and on the choice of $M$ (and hence $V$). But, excision implies that $\alpha_{\text{Dirac}}$ does not depend on $W$. 

13
Using this independence, we calculate

$$\alpha_{\text{Dirac}} = \text{ind } D(W \cup (\bar{V} \cup V)) \cup \cdots$$

$$+ \frac{1}{8} \sigma(W) - \frac{1}{16} \sigma(V)$$

$$= \text{ind } D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots)$$

$$+ \frac{1}{8} \sigma(W) - \frac{1}{16} \sigma(V)$$

$$= \text{ind } D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots)$$

$$+ \frac{1}{8} \sigma(W \cup \bar{V}) + \frac{1}{16} \sigma(V)$$

where in the last line we used that $$\sigma(W \cup \bar{V}) = \sigma(W) - \sigma(V)$$. Using excision, replace $$W \cup \bar{V}$$ by $$\bar{W}$$ to get

$$\alpha_{\text{Dirac}} = \text{ind } D(\bar{W} \cup (V \cup \bar{V}) \cup \cdots)$$

$$+ \frac{1}{8} \sigma(\bar{W}) + \frac{1}{16} \sigma(V) = -\alpha_{\text{Dirac}}$$

and we conclude that $$\alpha_{\text{Dirac}} = 0!$$
Finally, recall that the quaternionic nature of the Dirac operator implies (even on non-compact manifolds) that its index is even. So the mod 2 reduction of

$$\alpha_{\text{Dirac}} = \text{ind}(D) + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V)$$

is the Cappell-Shaneson invariant $\alpha$, which must then vanish as well (mod 2). □