Periodic-end Dirac Operators and Positive Scalar Curvature

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Recall from basic differential geometry: (X, g) Riemannian manifold $\Rightarrow$ Riemannian curvature tensor $\overset{\text{tr}}{\Rightarrow}$ scalar curvature S_g.

Question: *Which manifolds have a metric g with $S_g > 0$?*

We say that g is a metric of positive scalar curvature (PSC). Not all manifolds admit metrics with PSC, as can be shown by many techniques: Dirac operators (Lichnerowicz; Gromov-Lawson) and minimal surfaces (Schoen-Yau) in all dimensions; gauge theory (Seiberg-Witten) special to dimensions 3 and 4.

Will a describe a new technique to address this problem for some 4-manifolds, based on the analysis of the Dirac operator on non-compact manifolds.

I. Brief review of Dirac operators

Let X^n be an oriented Riemannian manifold with a spin structure. Recall that this means that $w_2(X) = 0$ and that there is a complex vector bundle $S \to X$, and a 'Clifford multiplication' $T^*X \otimes S \to S$. Some features of this:

- Clifford relation: if $\alpha_1, \alpha_2 \in T^*X$, and $s \in S$, $\alpha_1 \cdot (\alpha_2 \cdot s) + \alpha_2 \cdot (\alpha_1 \cdot s) = -2\langle \alpha_1, \alpha_2 \rangle s$.
- If $n = \dim X$ is even then $S \cong S^+ \oplus S^-$, with $T^*X \otimes S^\pm \to S^\mp$.
- If n is divisible by 4 then $S^\pm$ are quaternionic vector spaces, and Clifford multiplication is quaternionic linear.
- The metric on X gives a connection on S: for every tangent vector w on X, a derivation $\nabla_w : C^\infty(S) \to C^\infty(S)$.

Dirac operator $D : C^\infty(S) \to C^\infty(S)$:

$$Ds = \sum_i e_i \cdot \nabla_{e_i} s, \quad \{e_i\} \text{ an ON basis for } T_*X$$

Key facts when X is closed.

- $\dim \ker D$, $\dim \ker D^* < \infty$. In other words, D is a Fredholm operator.
- If $n = \dim X$ is even, then $D : C^\infty(S^\pm) \to C^\infty(S^\mp)$, ie $D = D^+ \oplus D^-$. We define $\operatorname{ind} D = \dim_{\mathbf{C}} \ker D^+ - \dim_{\mathbf{C}} \ker D^-$.
- Lichnerowicz-Weitzenböck formula:
$$D^* D = \nabla^* \nabla + \frac{1}{4} S_g$$
- The operator $\nabla^* \nabla$ is *non-negative*. Hence $S_g > 0 \Rightarrow \operatorname{ind} D = 0$.
- Likewise for Dirac twisted by flat bundle.

II. Periodic manifolds and Dirac operators.

Consider a closed manifold X with a map $f : X \to S^1$. This gives

- A $\mathbf{Z}$-cover $\tilde{X} \to X$, and lift $t : \tilde{X} \to \mathbf{R}$ of f.
- If X is spin, $\tilde{D}^+ : C^\infty(\tilde{S}^+) \to C^\infty(\tilde{S}^-)$.
- For any regular value $\theta \in S^1$ for f, a submanifold $f^{-1}\theta = M \subset X$.

Question: *Under what circumstances is $\tilde{D}^+$ a Fredholm operator?*

To make sense of this, need to complete $C^\infty(\tilde{S}^\pm)$ in some norm. Pick $\delta \in \mathbf{R}$, and define

$$L^2_\delta(\tilde{S}^\pm) = \{s \mid \int_{\tilde{X}} e^{t\delta}|s|^2 < \infty\}$$

Likewise, get Sobolev spaces $L^2_{k,\delta}(\tilde{S}^\pm)$.

Should really ask if the dimensions of the kernel/cokernel of $\tilde{D}^+ : L^2_{k,\delta}(\tilde{S}^\pm) \to L^2_{k-1,\delta}(\tilde{S}^\pm)$ are finite. If so we'll be sloppy and say $\tilde{D}^+$ is Fredholm on L^2_δ. The most useful case (for us) is $\delta = 0$.

Taubes' idea: Fourier-Laplace transform converts to family of problems on *compact* X. For each $c \in \mathbf{C}$, have the *twisted* Dirac operator $D_c : C^\infty(S) \to C^\infty(S)$ given by

$$D_c s = Ds + ic\, dt \cdot s.$$

Theorem 1. *(Taubes, 1987) Fix $\delta \in \mathbf{R}$. Suppose that $\ker D_c = \{0\}$ for all $c \in \mathbf{C}^*$ with $|c| = e^{\frac{\delta}{2}}$. Then $\tilde{D}^+$ is Fredholm on L^2_δ.*

Corollary 2. *If X has a Riemannian metric of positive scalar curvature, then $\tilde{D}^+$ is Fredholm on L^2_δ for any δ.*

This theorem, originally proved (more directly) by Gromov-Lawson (1983), is not *per se* an obstruction to existence of PSC metrics.

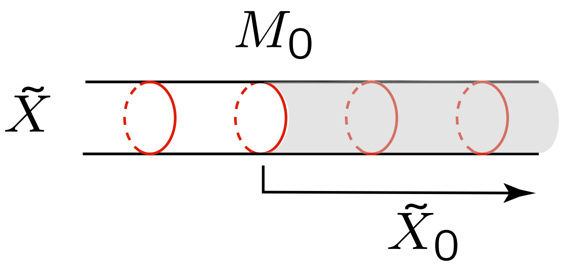
Theorem 3. *(R-Saveliev, 2006)*

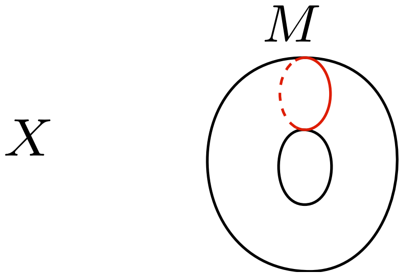
For a generic metric on X, the operator $\tilde{D}^+$ is Fredholm on L^2.

There are many manifolds (eg T^4) which admit no PSC metric, to which theorem 3 applies.

III. End-periodic manifolds.

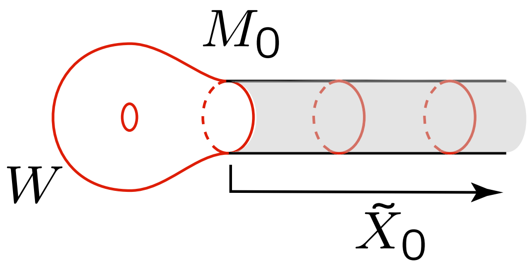
We are really interested in manifolds which are periodic in just one direction. Start with a $\mathbf{Z}$ covering $\tilde{X} \to X$, and consider a submanifold $M \subset X$ as pictured below. M does not separate X, and lifts to a compact submanifold $M_0 \subset \tilde{X}$.





Let $\tilde{X}_0$ be everything to the right of M_0, and choose a compact oriented spin manifold W with (oriented) boundary $-M$. From these pieces, form the *end-periodic manifold* with end modeled on $\tilde{X}$:

$$Z = W \cup_{M_0} \tilde{X}_0$$



Excision principle: Everything we said about Dirac operators on $\tilde{X}$ holds for Dirac operators on Z.

IV. Invariants of non-orientable 4-manifolds.

Let Y be the non-orientable S^3 bundle over S^1, a.k.a. $S^1 \times_\rho S^3$ where ρ is a reflection. More generally, let $Y_k = Y \#_k S^2 \times S^2$. Note that all of the Y_k admit a metric of PSC.

Question: *Is there a smooth manifold Y' homotopy equivalent to Y, but not diffeomorphic to Y?*

This is still unknown, but for $k > 0$, there are manifolds $Y'_k \simeq Y_k$ with $Y_k \not\cong Y_k$ constructed by Cappell-Shaneson, Akbulut, and Fintushel-Stern. We will use end-periodic Dirac operators to show that these exotic manifolds do not admit PSC metrics.

The difference between Y'_k and Y_k stems from Rohlin's theorem.

Rohlin's Theorem: Let X be a smooth, closed, oriented spin 4-manifold. Then the signature $\sigma(X)$ is divisible by 16.

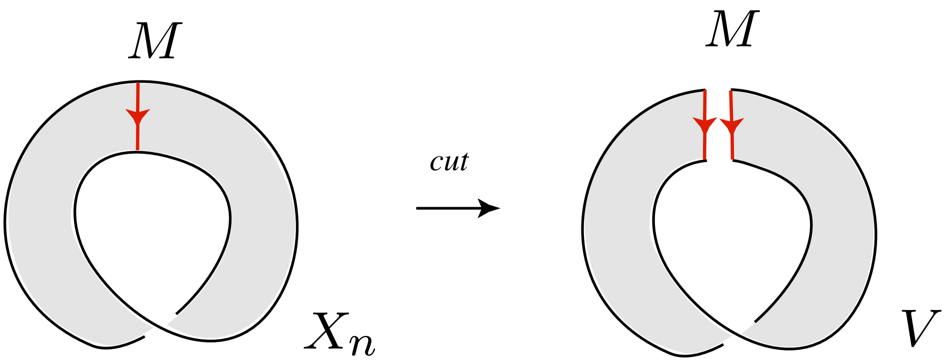
The fastest proof of this uses the index theorem (twice!) to show that the index of the Dirac operator is given by $\frac{\sigma(X)}{8}$. But as remarked earlier, for a 4-manifold that index is even, because the kernel and cokernel are *quaternionic* vector spaces.

Rohlin's theorem gives rise to the *Rohlin invariant* of a spin 3-manifold (M, s). Let $M = \partial W$ where the spin structure s extends over W. Then we define

$$\rho(M, s) = \frac{\sigma(X)}{8} \in \mathbf{Q}/2\mathbf{Z}.$$

Let's assume that we have a non-orientable manifold X_n with a map $f : X_n \to S^1$ such that $w_2(X_n) = 0$ and $w_1(X_n)$ is the pull-back of the generator of $H^1(S^1)$. As before, we get a submanifold $M = f^{-1}\theta$, and we can cut along M as before to get the *orientable* manifold $V = X_n - \text{nhd}(M)$.

Choose an orientation of V, then $\partial V = 2$ copies of M as shown below. It's not hard to show that in fact V has a spin structure, and so its boundary acquires one as well.



Following Cappell-Shaneson, define

$$\alpha(X_n) = \rho(M) - \frac{1}{16}\sigma(V) \in \mathbf{Q}/2\mathbf{Z}$$

which does not depend (up to sign) on choices made. For manifolds homotopy equivalent to Y_k, it turns out that $\alpha \equiv 0$ or 1 (mod $2\mathbf{Z}$). Cappell-Shaneson used a similar invariant to detect their exotic $\mathbf{RP}^4$.

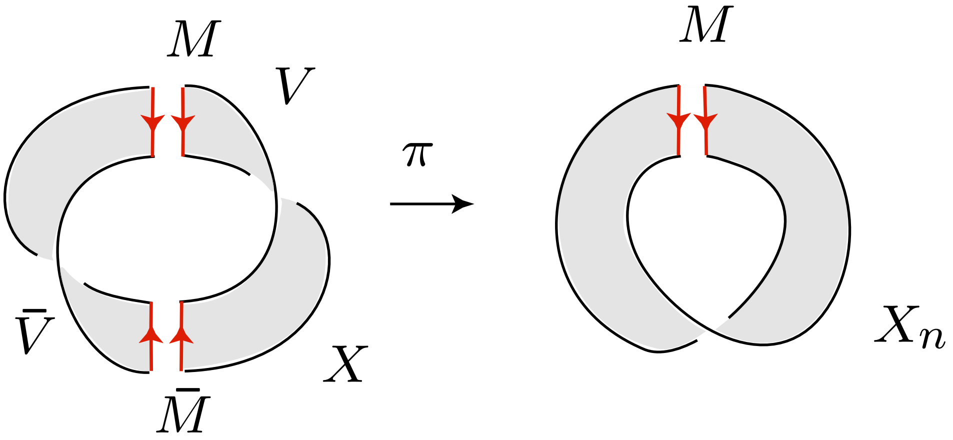
V. A new obstruction to existence of PSC metrics.

Theorem 4. *(R-Saveliev, 2006) Suppose that* $\alpha(X_n) \neq 0$*. Then* X_n *admits no metric of positive scalar curvature.*

Proof: Suppose that X_n does admit a PSC metric g_n. The idea is to use this to build a periodic-end manifold with positive scalar curvature on its end, and to use properties of the index of the Dirac operator to show that α must vanish. We continue with notation from before: M is a codimension-one submanifold of X_n, and V is X_n cut along M, with an orientation chosen.

First, consider the orientation double cover $\pi : X \to X_n$; note that X is canonically oriented. Since X is locally the same as X_n, the metric $g = \pi^* g_n$ has PSC. There are two lifts of V to X, but we can single one out by requiring that π preserve the orientation.

So we get the following picture



Now, choose a spin manifold W^4 with boundary M, and consider the periodic-end manifold (modeled on $\tilde{X} \to X$):

$$W \cup_M (\bar{V} \cup_{\bar{M}} V) \cup_M (\bar{V} \cup_{\bar{M}} V) \cup_M \cdots$$

Since g has PSC, the index of the Dirac operator on this manifold makes sense, and we define

$$\alpha_{\text{Dirac}} = \text{ind}\, D(W \cup (\bar{V} \cup V)) \cup \cdots) + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V)$$

This is not much of an invariant: it might depend on the choice of g_n, and on the choice of M (and hence V). But, excision implies that α_{Dirac} does <u>not</u> depend on W.

Using this independence, we calculate

$$\begin{aligned}\alpha_{\text{Dirac}} &= \text{ind}\, D(W \cup (\bar{V} \cup V)) \cup \cdots) \\ &\qquad + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V) \\ &= \text{ind}\, D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots) \\ &\qquad + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V) \\ &= \text{ind}\, D((W \cup \bar{V}) \cup (V \cup \bar{V}) \cup \cdots) \\ &\qquad + \frac{1}{8}\sigma(W \cup \bar{V}) + \frac{1}{16}\sigma(V)\end{aligned}$$

where in the last line we used that $\sigma(W \cup \bar{V}) = \sigma(W) - \sigma(V)$. Using excision, replace $W \cup \bar{V}$ by $\bar{W}$ to get

$$\begin{aligned}\alpha_{\text{Dirac}} &= \text{ind}\, D(\bar{W} \cup (V \cup \bar{V}) \cup \cdots) \\ &\qquad + \frac{1}{8}\sigma(\bar{W}) + \frac{1}{16}\sigma(V) = -\alpha_{\text{Dirac}}\end{aligned}$$

and we conclude that $\alpha_{\text{Dirac}} = 0$!

Finally, recall that the quaternionic nature of the Dirac operator implies (even on non-compact manifolds) that its index is even. So the mod 2 reduction of

$$\alpha_{\text{Dirac}} = \text{ind}(D) + \frac{1}{8}\sigma(W) - \frac{1}{16}\sigma(V)$$

is the Cappell-Shaneson invariant α, which must then vanish as well (mod 2). $\square$