Math 221b: Topics in Topology **Spring 2010**

Heegaard-Floer theory and applications

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The course will cover the basics of Heegaard-Floer theory and some of its many applications in low-dimensional topology. Heegaard-Floer theory is a package of homology groups associated to 3-dimensional manifolds. There are relative groups (for a knot in a 3-manifold) and maps induced by 4-dimensional cobordisms, giving the theory a structure (more or less) of a (3+1)-dimensional TQFT. The groups have a complicated algebraic structure as a module over the group ring $Z[U, U^{-1}]$, as well as various gradings that are important in applications.

My plan is to go through the construction of the groups, which involves a mix of classical 3-manifold topology (Heegaard splittings, handle slides, etc.) and the Floer theory of Lagrangian submanifolds in a symplectic manifold. The latter is a fundamentally analytic theory; I will gloss over most of the many technical details in this part. I will discuss the various flavors of the Heegaard-Floer theory; these go by names such as $HF^{\infty}, HF^{\pm}, \widehat{HF}$, and are related by exact sequences. I will outline the proof of invariance (from the many choices made in the definition) which involves the maps induced by 4-dimensional cobordisms.

We will do some simple but non-trivial calculations: for $S^2 \times S^1$ and lens spaces, for some surgeries on knots. The latter leads to the important idea of the relative groups, which go by the name of knot Floer homology.

Depending on the interest of the class, we can choose some of the applications to discuss. These include:

- Thurston norm on the homology of 3-manifolds.
- Genus bounds for 2-dimensional homology classes in 4-manifolds.
- Knot cobordism and the tau-invariant.
- Absolute gradings and the d-invariant.
- Relation to intersection forms of 4-manifolds.
- Applications to contact geometry of 3-manifolds.
- Combinatorial Heegaard-Floer theory.

References: There is as yet no textbook treatment of the subject. Most of what we will cover is from the fundamental papers of the creators of the theory, Peter Ozsvath and Zoltan Szabo. The foundations, including the analytic details, are in the first two papers mentioned below; the papers listed afterwards are mainly expository in nature. You can get a pretty good feel for the scope of the subject by searching for "Heegaard" and "Floer" on Mathscinet. All of the papers listed below are online (most are on the arxiv); some of the published versions have differences from the preprint versions.

Basic research articles

P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), **159** (2004), 1027–1158.

P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2), **159** (2004), 1159–1245.

P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants.* Adv. Math **186** (2004) 58-116.

Expository articles

P. Ozsváth and Z. Szabó, Heegaard diagrams and holomorphic disks. Different faces of geometry, 301–348, Int. Math. Ser. (N. Y.), 3, Kluwer/Plenum, New York, 2004.
http://arxiv.org/abs/math/0403029

P. Ozsváth and Z. Szabó, An introduction to Heegaard Floer homology. Floer homology, gauge theory, and low-dimensional topology, 3–27, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, 2006.
http://www.math.columbia.edu/∼petero/Introduction.pdf

P. Ozsváth and Z. Szabó, Lectures on Heegaard Floer homology. (English summary) Floer homology, gauge theory, and low-dimensional topology, 29–70, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, 2006.
http://www.math.columbia.edu/∼petero/Lectures.pdf