

## Review Problems for Final. Part I: problems from the book

The first half of this review sheet contains odd-numbered problems from the book. You may also want to consult the true/false review problems at the end of the chapters; we did 1-3 and 5 for the first two exams, but that still leaves chapters 6 and 7. I've suggested some from each of these chapters.

The final will cover the material we did in Chapters 1-3, (skip 4), 5 (except for 5.5), 6 (no Cramer's rule in 6.3), and 7 (in 7.4, only up through p. 334). I suggest that you do some reviewing of class notes, the exams and quizzes, the previous exam review sheets, and your homework. Don't forget that answers to HW and the old reviews are still available on the class web page.

**Section 1.1:** 9, 19, 29. (For this last one, compare with chapter 5.4, example 4.)

**Section 1.2:** 11, 27, 37, 43. (In number 27, think about how you should solve this using your knowledge of determinants.)

**Section 1.3:** 9, 23, 25, 35, 49, 55.

**Section 2.1:** 5, 7, 35.

**Section 2.2:** 5, 11 (we had 10 in HW 3, if it helps), 13, 15, 29, 41.

**Section 2.3:** 9, 13, 19, 29 (give two solutions: one involving reduced echelon form, the other involving determinants), 31, 43.

**Section 2.4:** 5,13, 15, odd problems in 17-25, 43, 45, 47.

**Section 3.1:** 11, 13, 19, 21, 25, 35,37, 45, 51.

**Section 3.2:** 7, 9, 11, 15, 19 (think before you calculate!), 25 (extra review credit: find an orthogonal basis), 27,29, 31,33, 39, 41.

**Section 3.3:** 5, 7, 9,15, 17, 21, 23, 27, 31, 37.

**Section 3.4:** 3, 7, 15, 33 (think about stuff from the last couple of days of class).

**Section 5.1:** 11, 19, 21, 25, 29, 31, 39 (in addition to the requested geometric argument, solve this using the formula for projections.)

**Section 5.2:** 5,7,13, 31, 35, 39. Don't worry about the QR factorization.

**Section 5.3:** 5, 7, 11, 17, 19, 21, 23, 25.

**Section 5.4:** 5, 9, 21, 23, 25, 37, 39.

**Section 6.1:** 7,11, 17, 25, 33, 35, 42-45, 47.

**Section 6.2:** 3, 7, 21, 27 (skew-symmetric means that  $A^T = -A$ ), 29, 37.

**Section 6.3:** 3, 7.

**Section 7.1:** 3, 5, 11, 17, 21, 33, 39.

**Section 7.2:** 3, 9, 13, 19, 23 (hint: consider  $\det(S^{-1}(\lambda I_n - A)S)$ ), 25 (from an earlier HW), 28 (refers to 25), 29.

**Section 7.3:** 9, 11, 13, 15, 21, 23, 33, 35, 37 (we did part (a), way back when), 43.

**Section 7.4:** 9, 13, 19, 25, 27, 31, 33, 37.

**Chapter 6 True/False:** Omit 30, 34, 36, 37, 41-44 (although they're kind of interesting!)

**Chapter 7 True/False:** 1-6, 8-14, 16, 18, 19 (hint: think about  $(S^{-1})^T$ ), 21, 24-37, 39-41, 46, 47, 48, 53, 56.

## Review Problems for Final. Part II: Additional problems.

These problems don't provide comprehensive coverage of the whole semester, but are intended to remind you how to work with some of the ideas that came up during the semester that seemed to cause some difficulties.

**Problem 1.** Suppose that the matrix  $B$  is derived from  $A$  by row operations. Why do the systems  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  have the same solutions? Is it true that for an arbitrary vector  $\vec{c}$ , the systems  $A\vec{x} = \vec{c}$  and  $B\vec{x} = \vec{c}$  have the same solutions? What is the relationship between the rank of  $A$  and that of  $B$ ? If  $A$  and  $B$  are square, what is the relationship between the determinants of  $A$  and  $B$ ?

**Problem 2.** (a) Find the matrix for the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  such that  $T(\vec{e}_1) =$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \text{ and } T(\vec{e}_3) = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

(b) Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ . Find the matrix for linear transformations  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $S(\vec{e}_1) = \vec{v}_1$  (etc.) and  $U(\vec{v}_1) = \vec{e}_1$ , etc.

(c) Find the matrix for a linear transformation  $V : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  with  $V(\vec{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ,  $V(\vec{v}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$ ,

$$\text{and } V(\vec{v}_3) = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

**Problem 3.** For these two problems, find a matrix with the properties described, or show that there is no such matrix. (a) A  $3 \times 3$  orthogonal matrix  $A$  with  $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ . (b) A  $3 \times 3$

orthogonal matrix  $B$  with  $B(\vec{e}_1) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$  and  $B(\vec{e}_2) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ . (Hint for (b): cross-product).

**Problem 4.** Consider a plane  $P$  in  $\mathbb{R}^3$ , and let  $\vec{u}_1, \vec{u}_2$  be an orthonormal basis for  $P$ .

(a) Show that for any  $\theta$ , the vectors  $\vec{w}_1 = \cos \theta \vec{u}_1 + \sin \theta \vec{u}_2$  and  $\vec{w}_2 = -\sin \theta \vec{u}_1 + \cos \theta \vec{u}_2$  form an orthonormal basis for  $P$ . Interpret geometrically.

(b) Find an ON basis  $\{\vec{u}_1, \vec{u}_2\}$  for the plane  $P$  spanned by  $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . For  $\theta = \pi/4$ , find the vectors  $\vec{w}_1, \vec{w}_2$  as in part (a).

(c) Using first the basis  $\{\vec{u}_1, \vec{u}_2\}$  and then the basis  $\{\vec{w}_1, \vec{w}_2\}$ , find the projection of  $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

into  $P$ . Explain the similarity or lack of similarity between your answers.

(d) Without calculating, describe the relationship between  $\vec{u}_1 \times \vec{u}_2$  and  $\vec{w}_1 \times \vec{w}_2$ . Check by calculation if you're not sure.

**Problem 5.** Which of the following is a subspace of  $\mathbb{R}^3$ ?

$A =$  all vectors of the form  $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  (where  $a$  and  $b$  run through all of the real numbers).

$B =$  all vectors of the form  $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

$C =$  all vectors of the form  $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

**Problem 6.** Suppose that  $P$  is a plane in  $\mathbb{R}^3$ , and that  $\vec{v}_1, \vec{v}_2$  is a basis for  $P$ . Let  $\vec{w}$  be a nonzero vector in  $P^\perp$  (the line perpendicular to  $P$ ). Show that  $\vec{v}_1, \vec{v}_2, \vec{w}$  is a basis for  $\mathbb{R}^3$ .