Solutions to review problems

Chapter 6 True/False: Omit 30, 34, 36, 37, 41-44

1. True. 2. True. 3. True; we did this for rows, but $\det A = \det A^T$.
4. False; the det gets multiplied by 4^4. 5. False; for example if both of A and B are I. 6.True: $(-1)^6 = 1$.
7. False; the columns are linearly dependent so the determinant is 0. 8. False; it's the other way around. 9. True. 10. True. 11. True; we did this in class. 12. False; it might be -1. 13. True.
14. False; the second and fourth column are linearly dependent.
15. False; if $k = -1$ or -2, there are two equal columns. 16. False; det $= -1$ since there are 3 inversions.
17. True; the pattern with the four 100's will give such a big number that there's no way for the determinant to come out 0. 18. False, take eg $A = 2I$. 19. True. 20. True; any nonzero noninvertible matrix will do. 21. True; fact 6.3.4 (think about the volume of a box with unit length sides.)
22. True; both are 0. (This might well be false for invertible matrices, if you had to rescale a row in converting to rref.) 24. False, for example $A = 2I_2$.
25. True, eg $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. The columns are orthogonal, so the absolute value of the determinant is the product of the lengths of the columns. (To get the sign straight, use fact 6.3.4).
26. False, eg $A = \begin{bmatrix} 8 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. 27. False; the formula gives the det after you switch two columns, so this is really $-\det A$. 28. True, eg $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. 29. False.
31. False (pick a diagonal matrix with big numbers on the diagonal.) 32. False, eg $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
33. False; how about $\pm I_2$? 35. True. If all of the $(n-1) \times (n-1)$ determinants are 0, then the Laplace expansion along the first column will give 0. 38. False; this would mean that $-1 = \det(-I_3)$ is a square (of $\det A$). 39. False; similar matrices have the same determinant. 40. False, for the same reason as the previous one.

Chapter 7 True/False: 1-6, 8-14, 16, 18, 19, 21, 24-37, 39-41, 46, 47, 48, 53, 56.

1. True. 2. True (by definition of the trace). 3. False; $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 but geometric multiplicity 1. 4. True; that's what diagonalizable means.
5. True. 6. True, with eigenvalue λ^3. 8. False (any odd-size matrix has at least one real eigenvalue).
9. True (we did this in class). 10. True. 11. False; same matrix as in 3.
12. False, for example let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 8 & 0 \\ 0 & 7 \end{bmatrix}$. Then $\alpha = 2$ is an eigenvalue of A, $\beta = 7$ is an eigenvalue of B, but $\alpha\beta = 14$ isn't an eigenvalue of AB. 13. True: if $A\vec{v} = 3\vec{v}$, then $A^2\vec{v} = AA\vec{v} = A3\vec{v} = 3A\vec{v} = 9\vec{v}$.
14. True; choose a basis for V and a basis for $V^\perp$. All the vectors in V (in particular the basis vectors) are eigenvectors with eigenvalue 1, and all of the basis vectors in $V^\perp$ are eigenvectors with eigenvalue 0.
16. False; any upper triangular matrix (other than a diagonal one) with diagonal elements 1 and -1. 18. True; the geometric multiplicity is the dimension of the kernel, which is $n-$ the rank.
19. True. If $S^{-1}AS = D$ where D is diagonal, then take the transpose of both sides, to get $S^T A^T (S^{-1})^T = D^T$. But $D^T = D$ is still diagonal, and $(S^{-1})^T = (S^T)^{-1}$, so A^T is also diagonalizable.
21. False: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ isn't diagonalizable (the only eigenvalue is 0, but the kernel is 1-dimensional) but A^2

is the 0-matrix.
24. False; let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.
25. True; take the inverse of both sides (as in 19). 26. False; it's the other way around.
27. True; the only way this is diagonalizable is if the eigenvalue 3 has geometric multiplicity three.
28. False; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$; then $A + B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. 29. False, for example the 0 matrix.
30. True. 31. No; eg the identity matrix. 32. True; they're both similar to the same diagonal matrix, hence to each other. 33. False, for example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. 34. False; $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
39. True (eigenvectors of distinct eigenvalues are always independent.)
40. True; an eigenvector of A is automatically an eigenvector for $A + 4I$, so the same eigenbasis will work.
41. False; a rotation by other than a multiple of π. 46. True; just multiply it on out. 47. True.
48. False; one of the eigenvalues might be 0, which would lower the rank. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
53. True; if the eigenvalue is 0, then $\vec{v}$ is in the kernel. If it's not 0, then $A(\frac{1}{\lambda}\vec{v}) = \vec{v}$ so $\vec{v}$ is in the image.
56. True; just compute: $(\vec{u}\vec{u}^T)\vec{u} = \vec{u}(\vec{u}^T\vec{u}) = \vec{u}(|\vec{u}|^2)$ so the eigenvalue is $|\vec{u}|^2$.

Part II. Additional Problems.

Problem 1. The first one goes back to the very first day of class; review your notes and come talk to me or ask at the review session! The second part is false; when we transform the matrix A by row operations, we must do the same row operations to $\vec{c}$ in order to preserve the equation $A\vec{x} = \vec{c}$. The ranks of A and B are the same, because they are both transformed into the same rref. If A and B are square, there's not much of a relation between their determinants, because you change the determinant if you multiply a row by a constant. The only relation is that $\det(A) = 0$ if and only if $\det(B) = 0$.

Problem 2. (a) The point of this problem is to remind you that the matrix is determined by the fact that the columns are just the vectors $T(\vec{e}_i)$. So the matrix is $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & -1 & 1 \\ 3 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

(b) As in (a), the matrix B for S is just the given vectors, written in columns: $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$. The matrix C for U is just the inverse of B, which (after some work) is given by $C = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$.

(c) V is the composition $T \circ U$, so its matrix is $AC = \begin{bmatrix} -3 & -4 & 8 \\ 6 & 1 & -5 \\ 5 & 1 & -3 \\ 10 & 4 & -10 \end{bmatrix}$.

Problem 3. (a) No such matrix exists; orthogonal matrices preserve lengths, but these two have different lengths ($\sqrt{2}$ and 2).
(b) We want the columns of B to form an orthonormal set of vectors. The first two columns of B are orthogonal to each other, and of length one, so we just need a 3d one. Take the third column (aka $B(\vec{e}_3)$) to be the cross-product $B(\vec{e}_1) \times B(\vec{e}_2) = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$.

Problem 4. (a) Compute $\vec{w}_1 \cdot \vec{w}_2 = (\cos\theta \vec{u}_1 + \sin\theta \vec{u}_2) \cdot (-\sin\theta \vec{u}_1 + \cos\theta \vec{u}_2) = -\cos\theta \sin\theta \vec{u}_1 \cdot \vec{u}_1 + \cos^2\theta \vec{u}_1 \cdot \vec{u}_2 + \sin^2\theta \vec{u}_1 \cdot \vec{u}_2 + \cos\theta \sin\theta \vec{u}_2 \cdot \vec{u}_2 = 0$, using $\vec{u}_1 \cdot \vec{u}_2 = 0$ and $\vec{u}_1 \cdot \vec{u}_1 = \vec{u}_2 \cdot \vec{u}_2 = 1$. Similarly, $\vec{w}_1 \cdot \vec{w}_1 = \vec{w}_2 \cdot \vec{w}_2 = 1$. Geometrically, we just rotated the orthonormal vectors $\vec{u}_1, \vec{u}_2$ by angle θ (in the plane they span) so it' s not so surprising that they're still ON.

(b) Use Gram-Schmidt; first replace $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1}{5} \\ \frac{2}{5} \end{bmatrix}$. Now normalize to get $\vec{u}_1 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} \frac{\sqrt{30}}{6} \\ \frac{-1}{\sqrt{30}} \\ \frac{\sqrt{30}}{15} \end{bmatrix}$. (Sorry, the numbers for these vectors were supposed to look better than this.) As I suggested in my email, I've left out the calculations for the rotation as they are a real mess.

(c) I'll do the projection with the u-basis; the projection with the w-basis, after a long struggle overcoming the evil square roots, comes out to be exactly the same.

$$\text{proj}_P \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \vec{u}_1 \right) \vec{u}_1 + \left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \vec{u}_2 \right) \vec{u}_2 = 0\vec{u}_1 + \frac{\sqrt{30}}{3} \vec{u}_2 = \begin{bmatrix} \frac{5}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}$$

(d) The two cross products are the same, which you can readily verify by waving your hands in the air. More formally, the two cross-products lie in the same line (i.e. $P^\perp$) and have the same length (1), using the formula $|\vec{x} \times \vec{y}| = |\vec{x}|\,|\vec{y}| \sin\alpha$ where the vectors make angle $\alpha = \pi/2$ in this case. Finally, they point in the same direction in the line $P^\perp$, because if you rotate $\vec{u}_1$ into $\vec{u}_2$ using your right hand, then you also rotate $\vec{w}_1$ into $\vec{w}_2$ using your right hand.

Problem 5. A is not a subspace, because $\vec{0}$ does not belong to A. This is not obvious! (In fact C looks similar, ie you don't immediately see that $\vec{0}$ is in C, but it is in fact a subspace.) To see it, you need to show that there is no choice of a, b that makes $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ equal to $\vec{0}$. This comes down to showing that there is no solution to the system of equations (in augmented matrix form)

$$\begin{bmatrix} 1 & 1 & \vdots & -3 \\ 2 & 1 & \vdots & 0 \\ 1 & 0 & \vdots & -1 \end{bmatrix}$$

which I leave in your expert hands.

B is a subspace, because it is the span of two vectors, which is always a subspace.

C is a subspace. To see this, note that $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So any vector in C can be written as $a' \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b' \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ where $a' = a+1$ and $b' = b+1$. In other words, C is also the span of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and is therefore a subspace. (In other other words, C is the same as B, which is a subspace.)