Solutions to Homework 14

Section 7.3.

Problem 6. Solving the equation $f_A(\lambda) = 0$ gives eigenvalues $\frac{7\pm\sqrt{57}}{2}$. The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 3 \\ \frac{4+\sqrt{57}}{2} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ \frac{4-\sqrt{57}}{2} \end{bmatrix}$.

Problem 8. The eigenvalues are $\lambda = 1, 2, 3$ with eigenbasis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Problem 14. The eigenvalues are 0 and 1, with 1 being a double root of the equation. When we solve for eigenvectors, we find that the eigenspace for 1 is 2-dimensional, and we get a basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Problem 18. The eigenvalues are $\lambda_1 = \lambda_2 = 0$ (ie 0 is a double root) and $\lambda_3 = \lambda_4 = 1$ (ie 1 is a double root). The eigenspace E_0 is 2-dimensional, with basis the vectors $\vec{e}_1, \vec{e}_3$. But the eigenspace E_1 is 1-dimensional, with basis $\vec{e}_2$. This is only 3 vectors, so there is no basis of eigenvectors.

Problem 22. We want A such that $A\vec{e}_1 = 7\vec{e}_1$ and $A\vec{e}_2 = 7\vec{e}_2$. The only such matrix is $A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

Problem 26. The constant term of $f_A(\lambda)$ is given by $(-1)^6 \det(A)$ which is negative by assumption. So $f_A(0) < 0$. On the other hand the leading term is λ^6, so $f_A(\lambda)$ is positive if λ is very large. So somewhere between 0 and very large, there must be a λ where the graph goes through the x-axis.

Problem 34. (a) Note that $SB = AS$, so if $\vec{x}$ is in the kernel of B, then $AS\vec{x} = SB\vec{x} = 0$, so $S\vec{x}$ is in the kernel of A.
(c) The book intended part (b) as an intermediate step to getting (c), but here it is done from first principles. By assumption, S is invertible. So if $\vec{x}_1, \ldots, \vec{x}_k$ is a basis for $\ker(B)$, then $S\vec{x}_1, \ldots, S\vec{x}_k$ are linearly independent elements of $\ker(A)$. Thus $\dim\ker(A) \geq \dim\ker(B)$. Now apply the same argument as in (a) to show that for any element $\vec{y}$ in $\ker(A)$, the vector $S^{-1}\vec{y}$ is in $\ker(B)$. Again, if $\vec{y}_1, \ldots, \vec{y}_l$ is a basis for $\ker(A)$, then $S^{-1}\vec{y}_1, \ldots, S^{-1}\vec{y}_l$ are linearly independent elements of $\ker(B)$. So $\dim(\ker(B)) \geq \dim(\ker(A))$. After all that we conclude that in fact $\dim(\ker(B)) = \dim(\ker(A))$, and so the ranks are the same (by rank + nullity theorem). In case you're worried, that's more involved than I'd intended, or would put on the exam.

Problem 44. a. $a_{11} = .7$ means that 70% of the pollutant in Lake Silvaplana is there a week later; some is carried down to Lake Sils by the river, and some is absorbed or evaporates. Similarly for the other diagonal entries. The matrix is lower triangular, since everything either evaporates or goes downstream.

b. The eigenvalues of A are 0.8, 0.6, 0.7, with corresponding eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The initial vector $\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is written with respect to this basis as $100\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

So, applying the matrix A t times gives $\vec{x}(t) = 100(0.8)^t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100(0.6)^t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100(0.7)^t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The levels of pollutant in each lake are the three components of this vector, ie $x_1(t) = 100(0.7)^t$, $x_2(t) =$

$-100(0.6)^t + 100(0.7)^t$, and $100(0.8)^t + 100(0.6)^t - 200(0.7)^t$. All of these eventually decrease exponentially towards 0.

Section 7.4.

Problem 2. Diagonalizable. The eigenvalues are 2,3, with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $S^{-1}AS$ is the diagonal matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Problem 12. Diagonalizable. The eigenvalues are 2,1,1 (double root) with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. If $S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $S^{-1}AS$ is the diagonal matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 16. Diagonalizable. The eigenvalues are 3,2,1 with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.