Solutions to Homework 10

Section 5.4

Problem 2. There are many ways to prove this. Here is what we did in class: we proved the identity $\vec{v} \cdot \vec{w} = \frac{1}{4}(|\vec{v} + \vec{w}|^2 - |\vec{v} - \vec{w}|^2)$. Note that if L preserves the lengths of vectors, then it also preserves the squares of those lengths. Using this twice (first for $L\vec{v} \cdot L\vec{w}$ and then for $\vec{v} \cdot \vec{w}$), we get

$$\begin{aligned} L(\vec{v}) \cdot L(\vec{w}) &= \frac{1}{4}(|L(\vec{v}) + L(\vec{w})|^2 - |L(\vec{v}) - L(\vec{w})|^2) \\ &= \frac{1}{4}(|L(\vec{v} + \vec{w})|^2 - |L(\vec{v} - \vec{w})|^2) \\ &= \frac{1}{4}(|\vec{v} + \vec{w}|^2 - |\vec{v} - \vec{w}|^2) = \vec{v} \cdot \vec{w} \end{aligned}$$

The middle line used the linearity of L. An alternate approach uses the result of problem 1 (which I did one day in class): Represent the transformation L by an orthogonal matrix A. Then $L(\vec{v}) \cdot L(\vec{w}) = (A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot (A^T A\vec{w})$ by problem 1. But A being orthogonal means $A^T A = I$, so the last quantity is just $\vec{v} \cdot \vec{w}$ again.

Problem 20. We need to solve the normal equation $A^T A\vec{x}^* = A^T\vec{b}$, which works out to

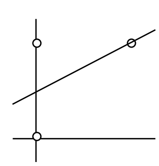
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

whose solution is $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. To check that $\vec{b} - A\vec{x}^*$ is orthogonal to the image of A, it suffices to show that $\vec{b} - A\vec{x}^*$ is orthogonal to each of the columns of A. So we compute $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, whose dot product with each column is 0.

Problem 30. We are looking for c_0, c_1 to satisfy the linear equations (from the given data): $c_0 + 0c_1 = 0$, $c_0 + 0c_1 = 1$, and $c_0 + 1c_1 = 1$. These are clearly inconsistent, so we solve them approximately for coefficients $\vec{x}^* = \begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix}$. Using $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, we get the normal equation

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

whose solution is $\vec{x}^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. So the best fitting line is $f(t) = \frac{1}{2} + \frac{1}{2}t$, drawn below; the method is set up to 'split the difference' vertically, as we see in the figure.



Problem 32. We want to find c_0, c_1, c_2 to satisfy the linear equations (from the given data): $c_0 = 27$, $c_0 + c_1 + c_2 = 0$, $c_0 + 2c_1 + 4c_2 = 0$, and $c_0 + 3c_1 + 9c_2 = 0$. We solve (approximately) for coefficients $\vec{x}^* = \begin{bmatrix} c_0^* \\ c_1^* \\ c_2^* \end{bmatrix}$. Using $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we get the normal equation

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 27 \\ 0 \\ 0 \end{bmatrix}$$

with solution $\vec{x}^* = \begin{bmatrix} \frac{513}{20} \\ \frac{-567}{20} \\ \frac{27}{4} \end{bmatrix}$. (I know, this is even worse than the problem on the handout.) Here is the parabola given by the equation $y = \frac{27}{4}t^2 - \frac{567}{20}t + \frac{513}{20}$, which is supposed to be the best fit. (Note that the axes are not at the same scale.) The original 4 points are drawn as dots on the graph.