Solutions to Homework 11

Section 6.1

Problem 4. $\det(A) = 8$ (the matrix is upper triangular).

Problem 12. $\det(A) = 0$ by direct calculation. (Of course, once we've done 6.2, this is easier, since we can do a row operation, not changing the determinant, to get a row of 0's.)

Problem 22. Not invertible since $\det = 0$.

Problem 28. We just treat λ as a constant, and compute $\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda - 2 & 3 \\ -3 & \lambda - 2 \end{bmatrix} = \lambda^2 - 4\lambda + 13$. Since the polynomial $\lambda^2 - 4\lambda + 13$ has no real roots $(16 - 52 < 0)$, there are no values of λ that make the determinant 0. Hence the matrix is invertible for all λ.

Section 6.2

Problem 2. $\det(A) = -45$.

Problem 8. Since $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, the only way that $T(\vec{x})$ can be $= \vec{0}$ is for $\vec{x}$ to be a linear combination of the $\vec{v}$'s. Hence the kernel of T is the span of $\vec{v}_2, \ldots, \vec{v}_n$, which is an $n - 1$ dimensional subspace of $\mathbb{R}^n$. The image of T is the real line $\mathbb{R}$ (since it must be 1-dimensional).

Problem 10. The determinant is -8, because we switch two rows (the first and last).

Problem 12. The determinant is still 8, because the matrix arises by a row addition.

Problem 16. (a) Multiply out the determinant to get $f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix} = (ab^2 - a^2 b) + (a^2 - b^2)t + (b - a)t^2$. The coefficient of t^2 is $b - a$.
(b) If $t = a$, then the first and third column are identical, so the determinant is 0. Likewise, if $t = b$, then the second and third columns are identical, so again the determinant is 0. Now, a polynomial of degree 2, with roots a and b must be of the form $k(t - a)(t - b)$, and we notice that k is just the coefficient of t^2. So the f we have (the determinant of that matrix) must be given by $(b - a)(t - a)(t - b)$.
(c) The matrix is invertible unless $t = a$ or b.
If you want to see the real point of this problem, then you should do problems 17-19.

Problem 24. If $\det(A) = 3$, then $\det(A^T) = 3$, and $\det(A^T A) = \det(A)\det(A^T) = 9$.

Problem 26. A is orthogonal if and only if $A^T A = I$, in which case $1 = \det(I_n) = \det(A^T A) = \det(A)\det(A^T) = \det(A)^2$. So $\det(A)$ must be ± 1.

Problem 39. (a) Note that $AA^{-1} = I$, so $\det(A)\det(A^{-1}) = 1$. Now if the entries in A and A^{-1} are integers, then both determinants are integers. But the only products of integers that give 1 are $1 \times 1 = 1$ or $-1 \times -1 = 1$. Hence $\det(A^{-1}) = \pm 1$. (Note that this part didn't use that these are 2×2 matrices.)
(b) Use the formula for A^{-1}, which involves dividing various entries of A by $\det(A)$. All of these give integers, if $\det(A) = \pm 1$.