Solutions to Homework 12

Section 6.3

Problem 2. The area is one-half of the area of the parallelogram spanned by those vectors, which is given by the absolute value of the determinant $\det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} = -50$. So the area of the triangle is 25.

Problem 14. You can use the formula (fact 6.3.7) for this volume $V = \sqrt{\det(A^T A)}$ where A has the given vectors as columns. We compute

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}, \quad \det(A^T A) = 6$$

so the volume is $\sqrt{6}$. Alternatively, we do the Gram-Schmidt process to the columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of A *without normalizing* to get an orthogonal basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$, and then the volume is $|\vec{w}_1|\,|\vec{w}_2|\,\vec{w}_3|$. Set $\vec{w}_1 = \vec{v}_1$. Then

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

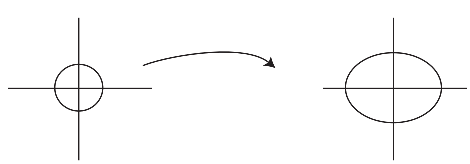
Finally,

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{w}_1 \cdot \vec{v}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{v}_3}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{9}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Now $|\vec{w}_1| = 1$, $|\vec{w}_2| = \sqrt{3}$, and $|\vec{w}_3| = \sqrt{2}$, again giving that the volume is $\sqrt{6}$. If you read the justification of fact 6.3.7, it is exactly this latter procedure, so it's no surprise that we got the same answer both ways.

Problem 16. False (although it would be true if this were a transformation from $\mathbb{R}^2$ to itself.) For example, let T be given by projection onto the xy plane, and let Ω be a parallelogram in the xy plane. Then the determinant is 0, but the area of Ω and that of $T(\Omega)$ are the same.

Problem 18. The area of an ellipse with semiminor axes a and b is given by πab. (This is a nice calculus problem.)
(a) The ellipse is drawn below; its axes are stretched by factors p and q, as indicated in the figure ($p = 2, q = 1.5$).



The area is πpq.
(b) The area gets multiplied by the absolute value of the determinant of A, so $|\det(A)|$ is the ratio of the area of the ellipse to the area of the unit circle, ie $(\pi pq)/\pi = pq$.
(c) The unit circle consists of all vectors of the form $\vec{x} = \cos(t)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin(t)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Its image is the

ellipse given by stretching the first vector by a factor of 4 and the second by a factor of 2, so it is given by all vectors of the form $\vec{x} = \cos(t)2\sqrt{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin(t)\sqrt{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.