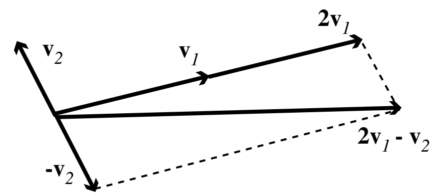
Solutions to Homework 4

Section 2.1

Problem 8. By solving the system $x_1+7y_2 = y_1,\ 3x_1+20x_2 = y_2$, we get $x_1 = -20y_1+7y_2,\ x+2 = 3y_1-y_2$.

Problem 16. A represents dilation (stretching) by a factor of 2. Its inverse is contraction by a factor of 1/2 so $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Problem 38. $\vec{v}_1$ and $\vec{v}_2$ are the columns of the matrix A with $T(\vec{x}) = A\vec{x}$. So $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2$. We can visualize this in this picture at right.



Section 2.2

Problem 2. The matrix is $\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

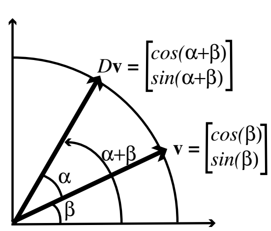
Problem 6. As we did in class, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{\vec{u}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\vec{u}\cdot\vec{u}}\vec{u}$. Using the given $\vec{u}$, we get $\begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}$.

Problem 10. Let $\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, so $\vec{u}\cdot\vec{u} = 25$. The first column of the matrix is given by $\text{proj}_L(\vec{e}_1) = \frac{4}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Likewise $\text{proj}_L(\vec{e}_2) = \frac{3}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. So the matrix is $\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$.

Problem 12. Again, the columns are $\text{proj}_L(\vec{e}_1)$ and $\text{proj}_L(\vec{e}_2)$; it's a little easier if $\vec{u}$ is a unit vector, so the term $\vec{u}\cdot\vec{u} = 1$. We get $\text{proj}_L(\vec{e}_1) = (\vec{u}\cdot\vec{e}_1)\vec{u} = u_1\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1^2 \\ u_1u_2 \end{bmatrix}$ and $\text{proj}_L(\vec{e}_2) = (\vec{u}\cdot\vec{e}_2)\vec{u} = u_2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1u_2 \\ u_2^2 \end{bmatrix}$. So the matrix for the projection is $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$.

Problem 20. $t(\vec{e}_1 = \vec{e}_1,\ T\vec{e}_2 = -\vec{e}_2,\ T\vec{e}_3 = \vec{e}_3$, so the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

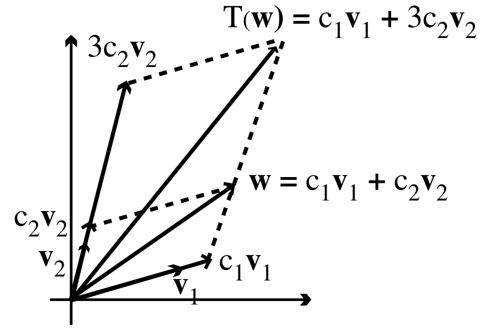
Problem 32. (a) Look at the figure



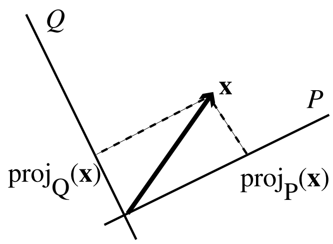
We see the coordinates for $D\vec{v}$ since the angle it makes with the x-axis is $\alpha + \beta$.

(b) Compute $D\vec{v} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}$. Comparing this with the coordinates that we got in part (a), we get the angle addition laws $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Problem 38. Write $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, ie $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. By linearity, $T(\vec{w}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1\vec{v}_1 + 3c_2\vec{v}_2$. The result is drawn at right.



Problem 40.



Visually, we see that $\vec{x} = \text{proj}_P(\vec{x}) + \text{proj}_Q(\vec{x})$.

Problem 42. The important thing to notice is that for any vector $\vec{v}$ on the line L, the projection $T(\vec{v})$ is just $\vec{v}$ again. So for any $\vec{x}$, we have $T(T(\vec{x})) = T(\vec{x})$, since the vector $T(\vec{x})$ lies on the line.