Solutions to Homework 5

Section 2.1

Problem 14. (a) By Exercise 13(a), the matrix is invertible if and only if $2k - 15 \neq 0$, i.e. if $k \neq 15/2$.
(b) This one is pretty hard, since we're not assuming that k is an integer. Assuming that $2k - 15 \neq 0$, some of the entries in the inverse matrix are $\frac{3}{2k-15}$, $\frac{2}{2k-15}$, and $\frac{k}{2k-15}$. Since the first two are integers, $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is an integer, say n. So $2k - 15 = \frac{1}{n}$, ie $k = \frac{15}{2} + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = \frac{15}{2}n + \frac{1}{2}$ is an integer as well, n must be odd. So we conclude that k must be of the form $\frac{15}{2} + \frac{1}{2n}$ where n is an odd integer. It is not hard to show that for every k of this form, all the entries in the inverse matrix are integers.

If you assume that k is an integer, then the problem is a little easier. In order for $\frac{3}{2k-15}$ to be an integer, the denominator has to divide the numerator, ie $2k - 15$ must be ± 1 or ± 3. Likewise, in order for $\frac{2}{2k-15}$ to be an integer , $2k - 15$ must be ± 1 or ± 2. So the only choice is that $2k - 15 = \pm 1$, in which case all the entries in the inverse matrix are actually integers.

Section 2.3

Problem 4. By doing row operations, we get $A^{-1} = \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$.

Problem 6. $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 18. We are supposed to write the x's in terms of the y's, which we do by examination: $x_1 = y_3$, $x_2 = y_1$, and $x_3 = y_2$.

Problem 34. Think about how you'd put this matrix in rref; you'll get the identity exactly when each of a, b, c is nonzero. In this case the inverse is $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

Problem 40. Suppose that columns i and j of the matrix A are equal. Then $A\vec{v} = \vec{0}$, where $\vec{v}$ is the column vector with 1 in the i^{th} spot and -1 in the j^{th} spot. Since there is a nonzero solution to $A\vec{x} = \vec{0}$, the matrix A cannot be invertible. Alternatively, if you apply row operations to A, it will continue to have two equal columns, so rref(A) will not be the identity matrix.

Section 2.4

Problem 4. $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$.

Problem 16. True: $(I_n - A)(I_n + A) = I_n^2 + A - A - A^2 = I_n - A^2$.

Problem 18. True: apply Fact 2.4.8 to $B = A$.

Problem 24. True: $I + A)(I + A^{-1}) = I^2 + A + A^{-1} + AA^{-1} = 2I + A + A^{-1}$.

Problem 28. Try $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Problem 40. $A = (AB)B^{-1} = ((AB)^{-1})^{-1}B^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$.

Problem 44. We want A such that $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, so we take $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$.