Solutions to Homework 7

3.3: For each of these, check consistency with fact 3.3.9 (rank + nullity theorem). 32, (35), 38, (39).

Section 3.2

Problem 2. Not a subspace; not closed under multiplication (eg by -1).

Problem 4. Yes, V = the span of those vectors is a subspace. If $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ and $\vec{w} = d_1\vec{v}_1 + \cdots + d_m\vec{v}_m$ are vectors in V, then $\vec{v} + \vec{w} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_m + d_m)\vec{v}_m$ is in V. Likewise, for any $k \in \mathbb{R}$, $k\vec{v} = (kc_1)\vec{v}_1 + \cdots + (kc_m)\vec{v}_m$ is in V.

Problem 6. a. Yes. To see this, we check the 3 conditions, using the fact that a vector $\vec{v}$ is in $V \cap W$ if and only if it is in V and in W. So, $\vec{0} \in V$ and $\vec{0} \in W$ (since they are both subspaces) implies that $\vec{0} \in V \cap W$. Likewise, if $\vec{v} \in V \cap W$, the and $\vec{w} \in V \cap W$, then $\vec{v} + \vec{w} \in V$ since V is a subspace, and $\vec{v} + \vec{w} \in W$ since W is a subspace, so $\vec{v} + \vec{w} \in V \cap W$. Finally, if $\vec{v} \in V \cap W$ and $c \in \mathbb{R}$, $c\vec{v} \in V$ and $c\vec{v} \in W$ (since they are both subspaces) so $c\vec{v} \in V \cap W$.
b. The union is in general not a subspace. For example, in $\mathbb{R}^2$, let $V = span(\vec{e}_1)$ and $W = span(\vec{e}_2)$. Then $\vec{e}_1 \in V \cup W$, and $\vec{e}_2 \in V \cup W$, but $\vec{e}_1 + \vec{e}_2 \notin V \cup W$ since it is neither a multiple of $\vec{e}_1$ nor of $\vec{e}_2$. So $V \cup W$ is not closed under vector addition.

Problem 14. Yes; compute the rref. (Really, what's going on is that the matrix with these as columns is upper triangular, with nonzero entries on the diagonal.)

Problem 18. Linearly dependent; rref is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 34. $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{0}$ means that $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}$. Solving this gives $\vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3$.

Problem 36. Since $\vec{v}_1, \ldots, \vec{v}_m$ are dependent, there is a non-trivial linear relation $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$. Taking T of both sides, and using the linearity of T, gives $c_1T(\vec{v}_1) + \cdots + c_mT(\vec{v}_m) = T(\vec{0}) = \vec{0}$. This is a non-trivial linear relation among the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$, so those vectors must be linearly dependent.

Problem 32. The second column is twice the first, so the first one alone is a basis. I actually meant this to be problem 42, whose answer follows.

Problem 42. Suppose that we have a linear relation $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots c_m\vec{v}_m = \vec{0}$. We want to show that all of the coefficients c_i are 0. To see this, take the dot product of both sides of the relation with the vector $\vec{v}_i$. Since $\vec{v}_j \cdot \vec{v}_i = 0$ if $j \neq i$ and $\vec{v}_i \cdot \vec{v}_i = 1$, all of the terms on the left are 0, except $c_i\vec{v}_i \cdot \vec{v}_i = c_i$. Since the right side is 0, we conclude that $c_i = 0$ for all i, so that the vectors are independent.

Section 3.3

Problem 6&16. The rref is I_3, so the kernel is $\{\vec{0}\}$, and the columns of A give a basis for the image. We have $rank = 3$, nullity $= 0$ and $n = 3$, confirming the rank + nullity theorem.

Problem 8&20. $rref(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The non-pivot columns (2, 4, and 5) give rise to the basis $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. The pivot columns (1 and 3) give columns 1 and 3 of A as the basis vectors for image(A). We have rank = 2 and nullity =3, and $n = 5$, again confirming rank+ nullity $= n$.

Problem 32. We are looking for all vectors whose dot products with the given two vectors are both 0. This is the same as the solution to $A\vec{x} = \vec{0}$, where A is the matrix $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. This matrix is already in rref, so we read off a basis for its kernel: $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

Problem 38. a. The rank could be any number from 0 to 3. (The rank is constrained to be less or equal to 3; we get all possible values from matrices in echelon form.) The dimension of the kernel is 5− rank, so it could be anything from 2 to 5.
b. The rank is less than or equal to 4, and could be anything from 0 to 4. For example, the 0-matrix has rank 0, the matrix $\begin{bmatrix} \vec{e}_1 & \vec{0} & \vec{0} & \vec{0} \end{bmatrix}$ has rank 1, $\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{0} & \vec{0} \end{bmatrix}$ has rank 2, $\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{0} \end{bmatrix}$ has rank 3, and $\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{bmatrix}$ has rank 4.