Solutions to Homework 8

Section 3.3

Problem 42. Take a basis of V, which consists of n elements since that is the dimension of V. Those elements are n linearly independent vectors in $\mathbb{R}^n$, and hence (see the summary at the end of the section) are a basis for $\mathbb{R}^n$. Thus the span of those vectors is V (because they're a basis for V) and also $\mathbb{R}^n$, so V and $\mathbb{R}^n$ are one and the same.

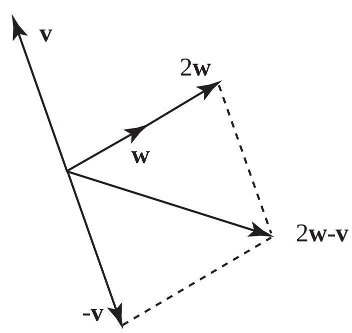
Section 3.4

Problem 2. Solve to find the given vector is 3 times the first basis vector plus 1 times the second. So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Problem 4. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

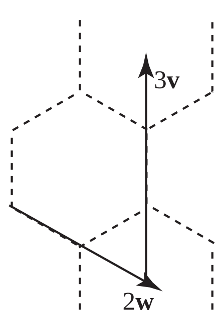
Problem 12. The coordinates are $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

Problem 22. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ means that $\vec{x} = -\vec{v} + 2\vec{w}$, which is illustrated below.



Problem 24. (a) $\vec{OP} = 2\vec{v} + \vec{w}$, so $[\vec{OP}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{OQ} = 2\vec{w} + \vec{v}$, so $[\vec{OP}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) $\vec{OR} = 3\vec{v} + 2\vec{w}$ is drawn below; the point R is clearly at the center of a tile.



(c) If the tip of some vector $\vec{u}$ is a vertex, then so is the tip of $\vec{u}+3\vec{v}$ and also of $\vec{u}+3\vec{w}$. (Draw a sketch to convince yourself of this.) We know (from part (a)) that the tip P of $2\vec{v}+\vec{w}$ is a vertex. Now we get from P to the endpoint S by adding $5(3\vec{v})$ and then $4(3\vec{w})$, so S must be at a vertex.

Section 5.1

Problem 2. $|\vec{v}| = \sqrt{29}$.

Problem 10. $\vec{u}\cdot\vec{v} = 6+3k$, which is 0 when $k = \frac{1}{2}$.

Problem 12. Using the hint,

$$\begin{aligned}|\vec{v}+\vec{w}|^2 &= (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \vec{v}\cdot\vec{v}+\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{v}+\vec{w}\cdot\vec{w}\\ &= |\vec{v}|^2+|\vec{w}|^2+2\vec{v}\cdot\vec{w}\\ &\le |\vec{v}|^2+|\vec{w}|^2+2|\vec{v}||\vec{w}| \quad (\mathrm{Cauchy-Schwarz})\\ &= (|\vec{v}|+|\vec{w}|)^2\end{aligned}$$

which is the inequality we were trying to show.

Problem 14. The horizontal components of $\vec{F_1}$ and $\vec{F_2}$ are $-|\vec{F_1}|\sin\beta$ and $|\vec{F_2}|\sin\alpha$, respectively. Since the system is at rest (and there is no horizontal force from the weight) the horizontal components must add up to 0, ie $-|\vec{F_1}|\sin\beta+|\vec{F_2}|\sin\alpha = 0$, or $\frac{|\vec{F_1}|}{|\vec{F_2}|} = \frac{\sin\alpha}{\sin\beta}$. To find the ratio of the lengths, note that $\overline{EA} = \overline{ED}\tan\alpha$ and $\overline{EB} = \overline{ED}\tan\beta$, so that $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan\alpha}{\tan\beta} = \frac{|\vec{F_1}|\cos\beta}{|\vec{F_2}|\cos\alpha}$. So except in the case that α and β are the same, Leonardo was wrong.

Problem 16. Note that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are already orthonormal. To find a fourth vector, let its components be x_1, x_2, x_3, x_4, and solve the equations $\vec{v}_1\cdot\vec{x} = \vec{v}_2\cdot\vec{x} = \vec{v}_3\cdot\vec{x} = 0$. We get that there is a 1-dimensional space of solutions, consisting of all multiples of the vector $\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$. Normalizing, we get $\vec{v}_4 = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$.

Problem 26. The two basis vectors are orthogonal, so $\mathrm{proj}_V\vec{x} = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1+\frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = 11\begin{bmatrix}2\\3\\6\end{bmatrix}-\begin{bmatrix}3\\-6\\2\end{bmatrix} = \begin{bmatrix}19\\39\\64\end{bmatrix}$.