Solutions to Homework 9

Section 5.2

Problem 6. If you follow through the Gram-Schmidt process, you will get the standard ON basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Problem 14. Start with $\vec{f_1} = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$. Then let $\vec{f_2} = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \text{proj}_{\vec{f_1}} \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \frac{100}{100} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Finally (before we normalize) let $\vec{f_3} = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - \text{proj}_{\vec{f_1}} \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - \text{proj}_{\vec{f_2}} \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - \frac{100}{100} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} - \frac{0}{100} \vec{f_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. These are orthogonal but don't have length 1. Normalizing, we get ON vectors $\frac{1}{10} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Problem 32. Solving the equations (the matrix is [1 1 1], which is in rref) gives a basis of solutions $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Applying Gram-Schmidt gives the ON basis $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

Problem 34. $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, so a basis of $\ker(A)$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Apply Gram-Schmidt to get an orthogonal basis $\vec{w}_1 = \vec{v}_1, \vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$. Dividing these by their lengths ($\sqrt{6}, \sqrt{30}$ respectively) gives an ON basis.

Section 5.3

Problem 2. Write $L(\vec{x}) = A\vec{x}$ where A is an orthogonal matrix. Use problem 1 (which I proved in class) to write $L(\vec{v}) \cdot L(\vec{w}) = (A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot (A^T A\vec{w})$. Since $A^T A = I_n$, this last term is just $\vec{v} \cdot \vec{w}$, as was to be proved.

Problem 4. We are assuming that $|A\vec{x}| = |\vec{x}|$ for all $\vec{x} \in \mathbb{R}^n$, where A is the matrix of the transformation. First, $\ker(A)$ is $\{\vec{0}\}$, since $A\vec{x} = 0 \Rightarrow |A\vec{x}| = 0 \Rightarrow |\vec{x}| = 0 \Rightarrow \vec{x} = \vec{0}$. By rank + nullity ($= n$) theorem, the rank (aka the dimension of the image) must be n. Since the rank is $\leq m$, we must have $n \leq m$.

As we proved in class, the fact that lengths are preserved implies the seemingly more general fact that dot products are preserved, or in other words $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}$ and $\vec{w}$. The columns of A are just the vectors $A\vec{e}_i$, which are ON since the vectors $\vec{e}_i$ are ON. The entries of A^TA are the dot products between columns of A, so $A^TA = I_n$. However, (if $m \neq n$) it will never be the case that AA^TI_m. For a silly example, let $n = 1$, $m = 2$, and let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then A preserves lengths, but $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (In general, although we didn't discuss this in class, the matrix AA^T represents projection onto Image(A).)

Problem 6. Yes, since if A is orthogonal, $A^T = A^{-1}$. But then $(A^T)^{-1} = (A^{-1})^T = (A^T)^T$, so A^T is orthogonal too.

Problem 10. Since the first two columns are orthogonal to the third, we must have $c = d = 0$. Then $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$ is an orthogonal matrix. Thus $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector in the plane, ie it is of the form $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some angle θ. The other column, $\begin{bmatrix} e \\ f \end{bmatrix}$ is a unit vector perpendicular to $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, so it is that vector rotated by $\pm\pi/2$. This gives either $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ (if rotated by $+\pi/2$) or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ (if rotated by $-\pi/2$). So the matrix must be

$$\begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ 0 & 0 & 1 \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta & 1 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{bmatrix}$$

Problem 16. Yes, since $(A^2)^T = (AA)^T = A^TA^T = AA = A^2$.

Problem 20. There are two ways to solve this problem: either by finding the projections of $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ onto the subspace, or by finding an ON basis and using fact 5.3.10. (We didn't study this latter one, so I was looking for the first of these.) First, find an orthogonal basis for W, say $\vec{w}_1 = \vec{v}_1$ and $\vec{w}_2 = \vec{v}_2 - \frac{8}{4}\vec{v}_1 = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$.

Then the columns of the matrix are $\text{proj}_W(\vec{e}_1) =$

$$\frac{1}{4}\vec{w}_1 + \frac{-1}{100}\vec{w}_2 = \begin{bmatrix} \frac{26}{100} \\ \frac{18}{100} \\ \frac{32}{100} \\ \frac{24}{100} \end{bmatrix}, \ \text{proj}_W(\vec{e}_2) = \frac{1}{4}\vec{w}_1 + \frac{7}{100}\vec{w}_2 = \begin{bmatrix} \frac{18}{100} \\ \frac{74}{100} \\ \frac{-24}{100} \\ \frac{32}{100} \end{bmatrix}, \ \text{proj}_W(\vec{e}_3) = \frac{1}{4}\vec{w}_1 + \frac{-7}{100}\vec{w}_2 = \begin{bmatrix} \frac{32}{100} \\ \frac{-24}{100} \\ \frac{74}{100} \\ \frac{18}{100} \end{bmatrix}$$

and $\text{proj}_W(\vec{e}_4) = \frac{1}{4}\vec{w}_1 + \frac{1}{100}\vec{w}_2 = \begin{bmatrix} \frac{24}{100} \\ \frac{32}{100} \\ \frac{18}{100} \\ \frac{26}{100} \end{bmatrix}$.

The other method (which we didn't study, but is nice anyway) would be to write down the matrix A whose columns are the vectors $\vec{w}_1, \vec{w}_2$ made into unit vectors by dividing by their respective lengths, ie $A = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.7 \\ 0.5 & -0.7 \\ 0.5 & 0.1 \end{bmatrix}$. Then the matrix is given by AA^T, which agrees with the answer given above.