# Solutions to review problems

## Chapter 1, Pages 37-38

**Problem 2.** F; consider the equation $x + y + z = 0$, repeated four times.

**Problem 3.** F; example 3a in section 1.3.

**Problem 8.** F; 0-matrix gives a counterexample.

**Problem 11.** F; the rank is 1.

**Problem 12.** F; the product on the left-hand side has two components.

**Problem 13.** T; Let $A = \begin{bmatrix} -3 & 0 \\ -5 & 0 \\ -7 & 0 \end{bmatrix}$ for example.

**Problem 14.** T; We have $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

**Problem 15.** T; the last component of the left-hand side is zero for all vectors $\vec{x}$.

**Problem 16.** T; $A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$ for example.

**Problem 21.** F; find the rref to see that the rank is always 2.

**Problem 23.** F; for example, let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,$\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Problem 26.** T; if $\vec{u} = a\vec{v} + b\vec{w}$ and $\vec{v} = c\vec{p} + d\vec{q} + e\vec{r}$, then $\vec{u} = ac\vec{p} + ad\vec{q} + ae\vec{r} + b\vec{w}$.

**Problem 27.** F; the system $x = 2, y = 3, x + y = 5$ has a unique solution.

**Problem 28.** F; Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

**Problem 29.** F; let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

**Problem 30.** T; by example 13 of section 1.3 (note that $\text{rank}(A) \leq 3$).

**Problem 31.** T; by example 3c of section 1.3, the equation $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$. So if $A(\vec{v} - \vec{w}) = \vec{0}$, $\vec{v} - \vec{w} = \vec{0}$ ie $\vec{v} = \vec{w}$.

**Problem 32.** T; note that rank(A) =4 by fact 1.3.4.

**Problem 34.** T; use rref to solve the system $A\vec{x} = \vec{0}$ and find $\vec{x} = \begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix}$. Let $t = 1$, to get $[\vec{u}\vec{v}\vec{w}] \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} = \vec{0}$, which says that $\vec{w} = 2\vec{u} + 3\vec{v}$.

**Problem 35.** F; Let $A = B = I_2$, for example.

## Chapter 2, Pages 94-95

**Problem 1.** T; the matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

**Problem 2.** F; the columns of a rotation matrix, being the rotated vectors $\vec{e}_1, \vec{e}_2$, are unit vectors.

**Problem 3.** T; fact 2.3.3.

**Problem 4.** T; let A=B in fact 2.4.8.

**Problem 5.** F; fact 2.4.3.

**Problem 6.** T; fact 2.4.9.

**Problem 7.** F; matrix AB will be $3 \times 5$.

**Problem 8.** F; a linear transformation must take $\vec{0}$ to $\vec{0}$.

**Problem 10.** T; fact 2.4.5.

**Problem 11.** F; fact 2.3.6, since det $= 0$.

**Problem 12.** T; compute its rref.

**Problem 13.** T; choose $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

**Problem 15.** T; simplify the polynomials to see that they're linear with 0 constant terms.

**Problem 18.** T; the columns are orthogonal and are unit vectors.

**Problem 19.** F; you actually multiply by 4.

**Problem 20.** T; the unique solution is $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$.

**Problem 24.** T; multiply it out. Alternatively, the left side is a shear of size k, repeated 3 times, while the right is a shear of size $3k$. These are the same.

**Problem 26.** T; try $A = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$.

**Problem 28.** F; if there were, then $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ would be invertible, but it's not.

**Problem 29.** F; If A has two identical rows, then so does $AB$ for any $B$, including any $B$ that might be $A^{-1}$. But $I$ doesn't have any identical rows.

**Problem 31.** F; $A$ might be a matrix representing rotation by $\frac{2\pi}{17}$.

**Problem 32.** F; reflections satisfy this property.

**Problem 33.** T; just divide all the rows by 5 and proceed with rref.

**Problem 34.** T; apply this to the unit vectors $\vec{e}_i$ to see that all of the columns of $A$ and $B$ must be equal.

**Problem 35.** T; since the matrices commute, you can rearrange this in any way you want, just like ordinary multiplication.

**Problem 36.** T; multiply both sides by $A^{-1}$.

**Problem 37.** F; try $A = I,\ B = -I$.

**Problem 40.** T; its inverse is $A(A^2)^{-1}$.

**Problem 42.** F; try $A =$ a shear, $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{e}_2$.

## Additional problems:

1. Give examples of $4 \times 5$ matrices, if possible, with ranks 0, 1, 2, 3, 4, and 5.

   Rank 0: the 0-matrix. Rank 1: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank 2: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank 3: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   Rank 4: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ There is no $4 \times 5$ matrix of rank 5; as we showed in class, the maximum rank is 4.

2. Let A be the $3 \times 4$ matrix $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 2 & 2 & 1 \\ -1 & 2 & -2 & -3 \end{bmatrix}$. Solve the equations $A\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Is there any choice of vector $\vec{b}$ that makes the equation $A\vec{x} = \vec{b}$ inconsistent?

   Let's solve all of these problems at the same time. Consider the equation $A\vec{x} = \vec{b}$ in matrix form: $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & b_1 \\ 3 & 2 & 2 & 1 & b_2 \\ -1 & 2 & -2 & -3 & b_3 \end{array}\right]$. After row operations, $A$ has been put in rref, and the equations have become

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -\frac{1}{2}\, b_1 + \frac{1}{2}\, b_2 \\ 0 & 1 & \frac{-1}{2} & -1 & \frac{3}{4}\, b_1 - \frac{1}{4}\, b_2 \\ 0 & 0 & 0 & 0 & -2\, b_1 + b_2 + b_3 \end{array}\right]$$

   Note that the rank is 2, and that there are 2 free variables ($x_3$, $x_4$). For $\vec{b} = \vec{0}$, the solution is $x_3 = s$, $x_4 = t$, $x_1 = -s - t$, $x_2 = \frac{s}{2} + t$. For $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, we get $x_3 = s$, $x_4 = t$, $x_1 = \frac{3}{2} - s - t$, $x_2 = -\frac{1}{4} + \frac{s}{2} + t$. Finally, we observe the bottom row of 0's in rref(A); if the corresponding constant term $-2\, b_1 + b_2 + b_3$ is nonzero, then there is no solution. (You might check that this term is actually 0 for the given $\vec{b}$.)

3. If the rank of a $5 \times 3$ matrix $A$ is 3, what is rref(A)? If the rank of a $4 \times 4$ matrix $B$ is 2, what are the possibilities for rref(B)?

   Since there are 3 leading entries in rref(A), rref(A) must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since there are 2 leading entries in rref(B), we know that the bottom two rows are all 0, and two of the columns must be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. These could be in columns 1&2, 1&3, 1&4, 2&3, 2&4, or 3&4. Here are the six corresponding forms of the matrices; the $*$'s indicate entries that are not necessarily 0.

$$\begin{bmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&*&0&*\\0&0&1&*\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&*&*&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&1&0&*\\0&0&1&*\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&1&*&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

4. Show that if $A$ and $B$ are $2\times 2$ matrices representing rotations by angles $\alpha$ and $\beta$ respectively, then $A$ and $B$ commute (i.e., $AB = BA$). Is this true if instead $A$ and $B$ represent reflections?

   There are several ways to solve this. One is just to write down both matrices, and multiply to get

$$AB = BA = \begin{bmatrix}\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{bmatrix}.$$

   The second is to observe that $AB$ represents a rotation by $\beta$ followed by a rotation by $\alpha$, which is the same as a rotation by $\alpha+\beta$, and hence is the same as a rotation by $\alpha$ followed by a rotation by $\beta$, otherwise known as $BA$. These two are essentially the same argument; you can see this by the addition rule for sin and cos.

5. Write down some reasonably big matrices, and multiply them. If you don't feel like making up your own examples, do a few from problems 1-15 on page 85.

6. Suppose that $A$ is a $2\times 2$ matrix that represents rotation by angle $\theta$, and $B$ is a $2\times 2$ matrix that represents reflection in a line $L$. Show, geometrically, that $ABA^{-1}$ represents reflection in the line $L'$, where $L'$ is $L$ rotated around by $\theta$.

   The figure shows a vector $\vec{v}$, to which we apply $A^{-1}$, then $B$ then $A$. First, $A^{-1}\vec{v}$ is $\vec{v}$ rotated by $-\theta$, so it makes the same angle, $\alpha$, with $L$ as $\vec{v}$ does with $L'$. Now apply $B$ to reflect $A^{-1}\vec{v}$ over the line $L$; note that the angle between the resulting vector and $L$ is still $\alpha$, but in the opposite direction. Apply the rotation $A$; it preserves angles, so that $ABA^{-1}\vec{v}$ makes an angle of $\alpha$ with $L'$, in the opposite direction from $\vec{v}$. Thus $ABA^{-1}\vec{v}$ is the reflection of $\vec{v}$ in $L'$.