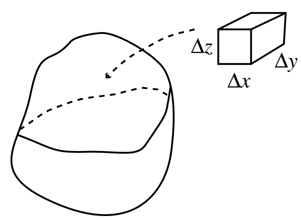
Notes on Triple integrals: Wednesday, November 26

These are some notes for my lecture on triple integrals. The idea of a triple integral is similar to the idea of a double integral. We are given some solid region E in 3-space, and a function $f(x, y, z)$, and we want to know 'how much of f is there in the region E'. A good example to think about is when f represents the density, say in mass/(unit volume) of a solid object E, and we want to know the the total mass. (We did something analogous for double integrals, where the function represents mass/(unit area).)

The idea, that should be familiar from our discussion of double integrals, is to break the region E into small cubes, whose sides are given by Δx, Δy, and Δz. The volume of such a small cube is given by $\Delta V = \Delta x \Delta y \Delta z$.



Assuming that the function f is continuous, ie doesn't vary too much over a small region, this small cube contributes roughly $f(x, y, z)\Delta V = f(x, y, z)\Delta x \Delta y \Delta z$ worth of mass. The total mass is approximately a sum of such terms; in the limit as the size of the cube goes to 0, the sum converges to what we call the triple integral of f over E, denoted

$$\iiint_E f(x, y, z)dV = \iiint_E f(x, y, z)dxdydz$$

If the function f is the constant function 1, then the integral represents the volume of the region E.

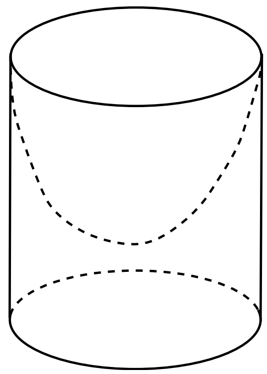
We want to learn how to calculate these triple integrals; the procedure is analogous to what we did with double integrals. The main complication is that 3-dimensional shapes are even more complicated that 2-dimensional ones. Let's start with the easiest kind, where E is a box $[a, b] \times [c, d] \times [e, f]$. Then we can express the integral as an iterated integral (and carry it out in any of the 6 possible orders).

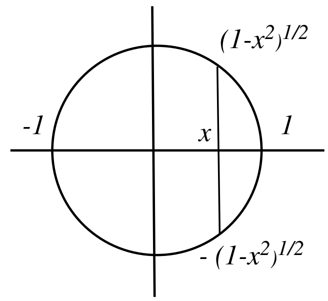
Example 1. Compute the integral of the function $f(x, y, z) = \frac{yz}{x}$ over the rectangular solid $1 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 2$, also known as $[1, 2] \times [0, 1] \times [0, 2]$, or in other words $\int_0^2 \int_0^1 \int_1^2 \frac{yz}{x} dx\, dy\, dz$. As the notation suggests, we integrate from inside out, and so do the x-integral, then the y-integral, and finally the z-integral.

$$\int_0^2 \int_0^1 \int_1^2 \frac{yz}{x} dx\, dy\, dz = \int_0^2 \int_0^1 [yz \ln x]_1^2\, dy\, dz = \ln 2 \int_0^2 \int_0^1 yz dy\, dz$$
$$= \ln 2 \int_0^2 y \left[\frac{z}{2}\right]_0^1 dy = \frac{\ln 2}{2} \int_0^2 y\, dy = \frac{\ln 2}{2} \left[\frac{y}{2}\right]_0^2 = \ln 2.$$

What about more complicated regions?

Example 2. Consider the solid S that lies underneath the paraboloid $z = 1 + x^2 + y^2$, above the xy–plane, and inside the cylinder $x^2 + y^2 = 1$. Let's find the integral over S of the function $f(x, y, z) = z$.





We need to describe the solid, in terms that we can use for the integral. Remember that we wrote the circle in the following way: $-1 \leq x \leq 1$, and for each x in that interval, y ranges over the interval $[-\sqrt{1-x^2}, \sqrt{1-x^2}]$. Now think about how to describe the solid in 3-dimensional terms: for each point (x, y) inside the circle, the height z can range from 0 to $1 + x^2 + y^2$. So we write the triple integral as

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1+x^2+y^2} z \; dz \; dy \; dx$$

This looks pretty complicated, so we'd better think about how to carry it out before we start integrating. The integral is set up to do the z-integral first; it's clear that we will get $z^2/2$, and evaluate with limits 0 and $1 + x^2 + y^2$. Thus we will be integrating some function of x and y (in fact it's $(1 + x^2 + y^2)^2/2$) over a region in the xy-plane. This is a double integral, of the sort that we've done before. In symbols, the integral is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\left[\frac{z^2}{2} \right]_0^{1+x^2+y^2} \right) \; dy \; dx = \frac{1}{2} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 + x^2 + y^2)^2 \; dy \; dx$$

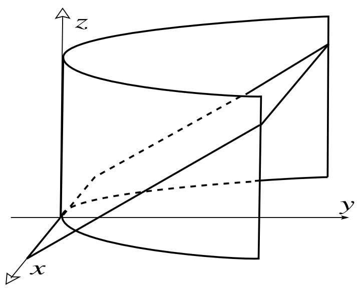
Let's pause again to think before we go any further. The next integral is the y-integral; to carry this out we should expand out $(1 + x^2 + y^2)^2$, getting a polynomial in y (remember x acts like a constant during this integration) that we can integrate. Then we'd stick in $\pm\sqrt{1-x^2}$ for limits, leaving us with something very messy to integrate. There's got to be a better way!

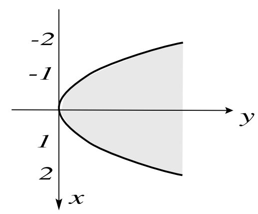
Well, of course there is. The double integral above is over a circular region in the plane, so we should immediately think about doing that integral using polar coordinates, rather than rectangular coordinates. (Remember the old saying about putting a square peg in a round hole?) In polar coordinates, the region over which we are integrating is described by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. The integrand $(1 + x^2 + y^2)^2$, when we substitute $x = r\cos\theta$, $y = r\sin\theta$, becomes $(1 + r^2)^2$. Finally, remember that in polar coordinates, the element of area $dA = dx \; dy$ becomes $rdr \; d\theta$. Putting this all together, the integral is

$$\frac{1}{2} \int_0^{2\pi} \int_0^1 (1 + r^2)^2 r dr \; d\theta = \frac{1}{2} \int_0^{2\pi} \left[\frac{1}{6}(1 + r^2)^3 \right]_0^1 d\theta = \frac{8}{12} \int_0^{2\pi} d\theta = \frac{4}{3}\pi$$

If there's time, we'll do one more example.

Example 3. Consider the solid bounded by the cylinder $y = x^2$, the xy-plane, the vertical plane $y = 4$, and the plane $z = y$, as drawn below. We want to calculate the integral of the function $f(x, y, z) = y$ over this solid. Again the main issue is setting it up. The solid sits over the region in the xy plane bounded by the parabola and $y = 4$.





This region could be described in two different ways, giving two different integrals. (In each of them, z will range from 0 to y.) The first is to say that x ranges from -2 to 2, and for each x value, y ranges from x^2 to 4. This leads to writing the integral as

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{y} y dz\ dy\ dx = \int_{-2}^{2}\int_{x^2}^{4} y^2\ dy\ dx = \frac{1}{3}\int_{-2}^{2}(64 - x^6)\ dx = \frac{1}{3}\left[64x - \frac{x^7}{7}\right]_{-2}^{2} = \frac{512}{7}.$$

The second way to write the integral is to say that y ranges from 0 to 4, and for each y value, x ranges between $-\sqrt{y}$ and $\sqrt{y}$. We would then write the integral as

$$\int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{y} y dz\ dx\ dy = \int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}} y^2\ dx\ dy = \int_{0}^{4} y^2\,[x]_{-\sqrt{y}}^{\sqrt{y}} = 2\int_{0}^{4} y^{5/2}\ dy = \frac{4}{7}\left[y^{7/2}\right]_{0}^{4} = \frac{512}{7}.$$