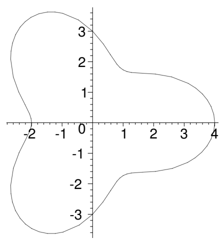
Solutions to final review problems

Solutions to problems from the book

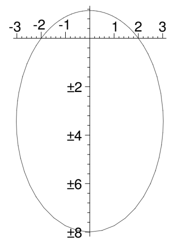
Chapter 11.

True–False. 2. False. 4. False (true in quadrants 1 and 4, false in quadrants 2 and 3). 6. True. 8. True. 10. True (sorry; we didn't do this stuff.)

Problem 10



Problem 14



Problem 16. $r = 2$.

Problem 20. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2-2t}{3t^2+6}$. So when $t = -1$, the slope is 4/9.

Problem 24. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{2t} = \frac{1}{2t} - \frac{3}{2}t$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\frac{3t^2+1}{4t^3}$$

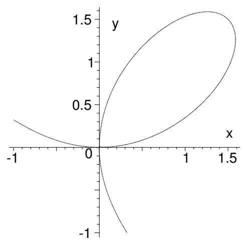
Problem 38. $L = \int_0^\pi \sqrt{\sin^6(\frac{\theta}{3}) + \sin^4(\frac{\theta}{3})\cos^2(\frac{\theta}{3})}\ d\theta = \int_0^\pi \sin^2(\frac{\theta}{3})\ d\theta = \left[\frac{1}{2}(\theta - \frac{3}{2}\sin(\frac{2\theta}{3}))\right]_0^\pi = \frac{\pi}{2} - \frac{3}{8}\sqrt{3}$.

Problems Plus

Problem 1. By the fundamental theorem of calculus, $dx/dt = \cos(t)/t$ and $dy/dt = \sin(t)/t$. The tangent is vertical when $dx/dt = 0$, which first happens at $t = \pi/2$. The length of the curve is

$$L = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\ dt = \int_1^{\pi/2} \frac{1}{t}\ dt = \ln(\pi/2)$$

Problem 5. (a) If you replace the parameter value t by $1/t$, then the values of x and y are interchanged. So if $(a,b) = (x(t), y(t))$, then $(b,a) = (x(1/t), y(1/t))$ is also on the curve. This doesn't work for $t = 0$, but $t = 0$ gives the point $(0,0)$ on the curve. (The symmetry is also easy to see if you do part (e).
(b) $dy/dt = \frac{6t-3t^4}{(1+t^3)^2}$ is 0 when $t = 0$ or $t = \sqrt[3]{2}$, so those values of t give horizontal tangents at $(0,0)$ and $(\sqrt[3]{4}, \sqrt[3]{2})$. From the symmetry of the curve (part (a)) we have vertical tangents at $(0,0)$ and $(\sqrt[3]{2}, \sqrt[3]{4})$.
(d)



(e) x^3+y^3 and $3xy$ both come out to $\frac{2t^3}{(1+t^3)^2}$.
(f) Use the equation from part (e) and substitute to get $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\sin\theta$, or $r = \frac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta} = \frac{3\sec\theta\tan\theta}{1+\tan^3\theta}$.
(g) The loop in question corresponds to $\theta \in (0, \pi/2)$, so the area is

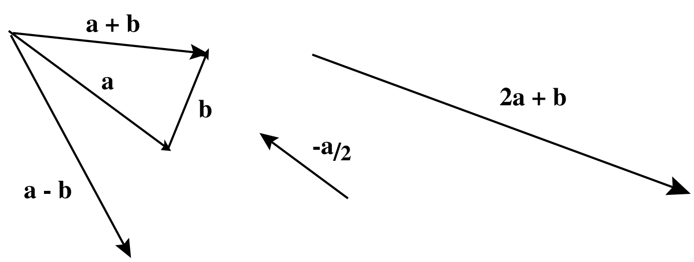
$$\int_0^{\pi/2} r^2/2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{3\sec^2\theta\tan^2\theta}{(1+\tan^3\theta)^2}\, d\theta = \frac{9}{2}\int_0^{\infty} \frac{u^2\, du}{(1+u^3)^2} = \frac{3}{2}$$

using the substitution $u = \tan\theta$.

Chapter 13.

True–False. 2 False. 4 True. 6 True. 8 False; for example $\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) \neq (\mathbf{i}\times\mathbf{i})\times\mathbf{j}$. 10 True. 12 False. 14 False.

Problem 2.



Problem 4. (a) $11\mathbf{i} - 4\mathbf{j} - \mathbf{k}$. (b) $\sqrt{14}$. (c) -1. (d) $-3\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$. (e) $4\sqrt{35}$. (f) 18. (g) $\mathbf{0}$. (h) $33\mathbf{i} - 21\mathbf{j} + 6\mathbf{k}$. (i) $-1/\sqrt{6}$. (j) $-\frac{1}{6}(\mathbf{i}+\mathbf{j}-2\mathbf{k})$. (k) $\cos\theta = \frac{-1}{2\sqrt{21}}$ so $\theta \approx 96°$.

Problem 10. $\vec{AB} = \langle 1, 3, -1\rangle$, $\vec{AC} = \langle -2, 1, 3\rangle$ and $\vec{AD} = \langle -1, 3, 1\rangle$. The volume is the (absolute value of) $\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = |-6| = 6$.

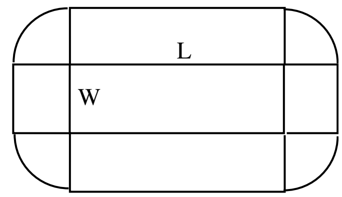
Problem 24. (a) The normal vectors are not parallel (by direct examination), so the planes are not parallel. They are not perpendicular (their dot product is -5) so the planes are not perpendicular. (b) The angle between the planes is the angle between the normal vectors, or $\cos^{-1}(\frac{-5}{\sqrt{87}}) \approx 122°$. Actually, we usually use the angle less than $90°$, so we should say $58°$.

Problem 34. Complete the square to get $x = (y-1)^2 + (z-2)^2$, which is a circular paraboloid opening up in the positive x direction.

Problem 46. Cylindrical: $r^2 + z^2 = 2r\cos\theta$. Spherical : $\rho^2 = 2\rho\sin\phi\cos\theta$ or $\rho = 2\sin\phi\cos\theta$

Problems Plus

Problem 2. First do it in 2 dimensions. We have a rectangle of size $L \times W$, two rectangles of size $L \times 1$, two of size $1 \times W$, plus four quarter circles of radius 1. In three dimensions, the central part is a rectangular solid of dimensions $L \times W \times H$. In addition we have rectangular solids on the sides of dimensions $L \times W \times 1$, $L \times 1 \times H$, and $1 \times W \times H$.



The corners are made of four quarter cylinders of radius 1 and height H, four of height L, and four of height W. Finally, the corners get 8 eighths of a sphere of radius 1. All this adds up to $LWH + 2LW + 2WH + 2LH + \pi 1^2 W + \pi 1^2 L + \pi 1^2 H + \frac{4}{3}\pi 1^3 = LWH + 2(LW + WH + LH) + \pi(L + W + H) + \frac{4}{3}\pi$.

Problem 4. (a) Let $\mathbf{v}_c = 180\mathbf{j}$ be the velocity shown by the compass, let $\mathbf{w}$ be the velocity of the wind, and let $\mathbf{v}_g = \mathbf{v}_c + \mathbf{w}$ be the actual velocity relative to the ground. Since the plane flew 80km in a half-hour, $|\mathbf{v}_g| = 160$. So $\mathbf{v}_g = (160\cos 85°)\mathbf{i} + (160\sin 85°)\mathbf{j} \approx 13.9\mathbf{i} + 159.4\mathbf{j}$. So $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_c \approx 13.9\mathbf{i} - 20.6\mathbf{j}$.
(b) The pilot should correct course by traveling with velocity $\mathbf{v}_c - \mathbf{w} \approx -13.9\mathbf{i} + 200.6\mathbf{j}$.

Chapter 14.

True–False. 2 True. 4 True. 6 False.

Problem 4. The point corresponds to parameter value $t = 1$. The velocity vector is $\langle 2, 4, 3\rangle$, so the equation of the line is $\mathbf{x}(t) = (1 + 2t, 1 + 4t, 1 + 3t)$.

Problem 6. (a) We want $y = 0$, ie $t = 1/2$, so the point is $(15/8, 0, -\ln 2)$. (b) The tangent vector is $\mathbf{r}'(1) = \langle -3, 2, 1\rangle$. So the tangent line (in vector form) is $\mathbf{x}(t) = (1 - 3t, 1 + 2t, t)$.

Problems Plus

Problem 1. (a, b) $\mathbf{v}(t) = \mathbf{r}'(t) = R\omega\cos\omega t\mathbf{i} - R\omega\sin\omega t\mathbf{j}$ and direct computation shows $\mathbf{r}\cdot\mathbf{r}' = 0$ and $|\mathbf{v}| = R\omega$. The time of revolution is the circumference of the circle divided by the speed, ie $T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}$.
(c) $\mathbf{a}(t) = -R\omega^2\sin\omega t\mathbf{i} - R\omega^2\cos\omega t\mathbf{j} = -\omega^2\mathbf{r}(t)$ which clearly points inward.
(d) Use Newton's law $\mathbf{F} = m\mathbf{a}$; then $|\mathbf{F}| = m|\mathbf{a}| = mR\omega^2 = \frac{mR^2\omega^2}{R} = \frac{m|\mathbf{v}|^2}{R}$.

Problem 3. (a) Taking the given formulas for granted, the maximum height is achieved when the $\mathbf{j}$-component of the velocity is 0. This occurs when $v_0\sin\alpha = gt$, ie at $t = \frac{v_0\sin\alpha}{g}$. The height at this time is given by $y = \frac{v_0^2\sin^2\alpha}{2g}$. As a function of α, this is maximized when $\alpha = \pi/2$, so that the maximum height is $\frac{v_0^2}{2g}$.

Problem 5(a). Divide by m, and take the integral of both sides to give

$$\int \frac{d^2\mathbf{R}}{dt^2}\,dt + \frac{k}{m}\int \frac{d\mathbf{R}}{dt}\,dt = -g\int \mathbf{j}\,dt + \mathbf{C} \quad \Rightarrow$$
$$\frac{d\mathbf{R}}{dt} + \frac{k}{m}\mathbf{R} = -gt\mathbf{j} + \mathbf{C}$$

Plugging at $t = 0$ gives $\mathbf{C} = \mathbf{v}(0)$. (The problem assumes that $\mathbf{R}(0) = \vec{0}$.)

Chapter 15.

True–False. 2 False (Clairaut). 4 True. 6 False. 8 False. 10 True. 12 False (see exercise 15.7.35).

Problem 6. Parabolas opening down.

Problem 12. We estimate $T_x(6,4)$ by averaging the rates of change for $\Delta x = \pm 2$, which are $\frac{T(8,4)-T(6,4)}{2} = 3$ and $\frac{T(4,4)-T(6,4)}{-2} = 4$ respectively, so $T_x(6,4) \approx 3.5$. Similarly, $T_y(6,4) \approx -3$. So a linear approximation is $L(x,y) = 80 + 3.5(x-6) - 3(y-4)$. This gives an approximation (to the actual value of $T(5,3.8)$) of $L(5,3.8) = 80 - 3.5 - 3(-0.2) = 77.1°C$.

Problem 14. $u_r = -e^{-r}\sin 2\theta$. $u_\theta = 2e^{-r}\cos 2\theta$.

Problem 20. $z_{xx} = 0$, $z_{yy} = 4xe^{-2y}$, and $z_{xy} = z_{yx} = -2e^{-2y}$.

Problem 26. $z_x = 1$ and $z_y = 0$, so an equation of the tangent plane is $z - 1 = x$.
(b) A normal vector to the tangent plane is $\langle 1, 0, -1\rangle$, so parametric equations for the normal line are $x = t,\ y = 0,\ z = 1 - t$.

Problem 34. (a) $dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = \frac{1}{2}ydx + \frac{1}{2}xdy$.Since $|\Delta x| \le .002m$ and $|\Delta y| \le .002m$ (mind your units!) the maximum error would be about $6(.002) + \frac{5}{2}(.002) = .017m^2$.
(b) For the hypotenuse, h, we have $dh = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$. So the maximum error is about $dh = \frac{5}{13}(.002) + \frac{12}{13}(.002) \approx .0026m$.

Problem 40. $A = \frac{1}{2}\sin\theta$, so by the chain rule, $\frac{dA}{dt} = \frac{1}{2}\left[(y\sin\theta)\frac{dx}{dt} + (x\sin\theta)\frac{dy}{dt} + (xy\cos\theta)\frac{d\theta}{dt}\right]$. For the given values, this works out to $\approx 60.8 in^2/s$.

Problem 50. From problem N, we want the normal to both surfaces. One is given by $\mathbf{N} = \nabla(z - 2x^2 + y^2) = \langle 8, 4, 1\rangle$ and the other by $\mathbf{n} = \langle 0, 0, 1\rangle$. So a tangent vector is given by $\mathbf{N}\times\mathbf{n} = 4\mathbf{i} - 8\mathbf{j}$. (You can figure this out without problem N, by saying that the tangent vector is perpendicular to the normal vector to the curve $2x^2 + y^2 = 4$ in the plane $z = 4$.) Hence parametric equations are given by $x = -2 + 4t,\ y = 2 - 8t,\ z = 4$.

Problem 52. $f_x = 3x^2 - 6y$, $f_y = -6x + 24y^2$. Solving $f_x = 0$ gives $y = \frac{1}{2}x^2$, and then substituting into $f_y = 0$ gives $6x(x^3 - 1) = 0$. The critical points are then $(0,0)$ and $(1, \frac{1}{2})$. The second derivative test shows the first is a saddle, and the second is a local minimum.

Problem 54. $f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2 + x^2 + y)/2$. So the only critical point is $(0,-2)$, which is a local minimum by the second derivative test.

Problems Plus

Problem 4. Let's show something less than was asked for, namely that the function is not continuous if $r \leq 2$. To do this, we need to show that the limit as $(x, y, z) \to (0, 0, 0)$ either does not exist, or is not 0 if it does exist. Consider the function along the line $(x, 0, 0)$, ie $g(x) = \frac{x^r}{x^2}$. If $r = 2$, this has limit 1, but that is not the value of the function at $(0, 0, 0)$, so the function is not continuous. If $r < 2$, then the limit certainly doesn't exist, so again the function is not continuous. (It turns out to be continuous if $r > 2$, but that takes considerable work to prove.)

Problem 7. There's nothing else to do but plug in the expressions for x, y, and z in cylindrical (part a) or spherical (b) coordinates and grind away with the chain rule. I'll check your work if you like.

Chapter 16.

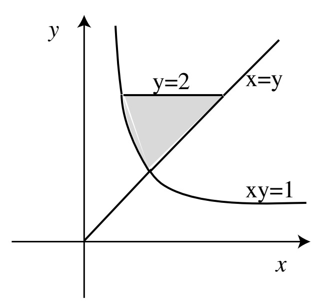
True–False. 2 False. If you change the order of integration, the limits will be $\int_0^1 \int_y^1$. 4. True (use the fact that $e^{y^2} \sin y$ is an odd function). 6. True: The integrand is ≤ 3, and the area of the base is 3. (We didn't actually discuss this in class, but it should make intuitive sense.)

Problem 4. $\int_0^1 \int_0^1 ye^{xy} dxdy = \int_0^1 (e^y - 1)dy = e - 2$.

Problem 14. The region is to the right of the curve $x = \sqrt{y}$ with $0 \leq y \leq 1$. This can also be described as $0 \leq x \leq 1$, and below the curve $y = x^2$. So the integral is $\int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dy\ dx = \int_0^1 \frac{1}{2} xe^{x^2} dx = \frac{1}{4} \left[e^{x^2} \right]_0^1 = \frac{1}{4}(e - 1)$.

Problem 20. A picture would help.

$$\int_1^2 \int_{1/y}^y y \quad dx \quad dy \quad = \quad \int_1^2 y \left(y - \frac{1}{y} \right) dy \quad =$$
$$\int_1^(y^2 - 1)dy = \frac{4}{3}.$$



Problem 22. Use polar coordinates: $\int_0^{\pi/2} \int_1^{\sqrt{2}} (r \cos \theta) r dr d\theta = \frac{1}{3}(2^{3/2} - 1)$.

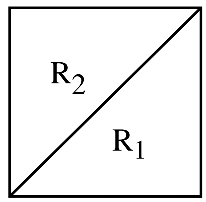
Problem 32. z ranges from 0 to $3 - y = r - r \sin \theta$. So the volume is

$$\int_0^{2\pi} \int_0^2 \int_0^{2 - r \sin \theta} r dz dr d\theta = \int_0^{2\pi} \int_0^2 (3r - r^2 \sin \theta) dr d\theta = 12\pi.$$

Problems Plus

Problem 2. Divide the rectangle into two pieces, R_1 and R_2, according to whether x or y is larger.

(This is the same as whether x^2 or y^2 is larger.) The integral splits accordingly, with the function being e^{x^2} on R_1 and e^{y^2} on R_2 . The integral over R_1 is $\int_0^1 \int_0^x e^{x^2} dydx$, while the integral over R_2 is $\int_0^1 \int_0^y e^{y^2} dxdy$. The first one gives $\int_0^1 xe^{x^2} dx = \frac{1}{2}(e-1)$; the second gives $\int_0^1 ye^{y^2} dy = \frac{1}{2}(e-1)$ so the total is $(e-1)$.



Chapter 17.

Problem 2. Use the parameterization $x = t,\ y = t^2$ for $t \in [0,1]$. We get $ds = \sqrt{1+4t^2}dt$ so the integral becomes

$$\int_0^1 t\sqrt{1+4t^2}dt = \frac{2}{3}\frac{1}{8}\left[(1+4t^2)^{3/2}\right]_0^1 = \frac{1}{12}(5^{3/2}-1)$$

Problem 4. Set $x = t$ and $y = \sin t$. Then (use integration by parts for $\int t \sin t\ dt$)

$$\int_C xy\ dx + y\ dy = \int_0^{\pi/2} t\sin t\ dt + \sin t \cos t\ dt = [-t\cos t + \sin t + \frac{1}{2}\sin^2(t)]_0^{\pi/2} = \frac{3}{2}$$

Problem 6. $\displaystyle\int_C \sqrt{xy}\ dx + e^y\ dy + xz\ dz = \int_0^1 \sqrt{t^6}\, 4t^3\ dt + e^{t^2}\, 2t\, dt + t^7\, 3t^2\ dt = \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10}\right]_0^1 = \frac{4}{7} + e - 1 + \frac{3}{10}.$

Problem 8. First compute $\mathbf{r}'(t) = \cos t\mathbf{i} + \mathbf{j}$. Then (again with integrating by parts)

$$\int_C ((1+t)\sin t\mathbf{i} + \sin^2 t\mathbf{j}) \cdot (\cos t\mathbf{i} + \mathbf{j})\ dt = \int_0^\pi (1+t)\sin t\cos t\ dt + \sin^2 t\ dt =$$

$$\frac{1}{2}\int_0^\pi (1+t)\sin 2t; dt + (1-\cos 2t)\ dt = \frac{1}{2}\left[-\frac{1}{2}(1+t)\cos 2t + \frac{1}{4}\sin 2t + t - \frac{1}{2}\sin 2t\right]_0^\pi = \frac{\pi}{4}.$$

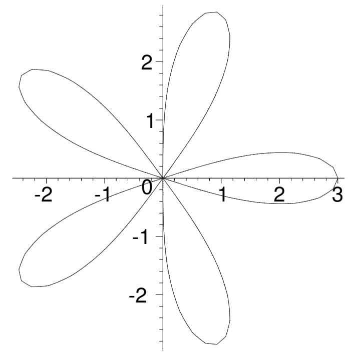
Problem 10. The problem asks for the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ for two paths $C = C_1$ or C_2. Let C_1 be the straight line, where $\mathbf{r}_1(t) = (1-t)(3,0,0) + t(0,\pi/2,3)$ and let C_2 be the spiral where $\mathbf{r}_2(t)$ is given in the book. For the first integral, $x = 3(1-t)$, $y = \pi/2t$ and $z = 3t$, so the integral

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 = \int_0^1 (3t)(-3)dt + (3-3t)(\pi/2)dt + (\pi/2t)(3)dt = \left[-\frac{9}{2}t^2 + \frac{3\pi}{2}t - \frac{3\pi}{4}t^2 + \frac{3\pi}{4}t^2\right]_0^1 = -\frac{9}{2} + \frac{3\pi}{2}.$$

For the second integral, use the given parameterization (and note that $t \in [0,\pi/2]$):

$$\int_0^{\pi/2} (-9\sin^2 t + 3\cos t + 3t\cos t)dt = \left[-\frac{9}{2} + \frac{9}{4}\sin 2t + 3\sin t + 3(t\sin t - \sin t)\right]_0^{\pi/2} = -\frac{3\pi}{4}.$$

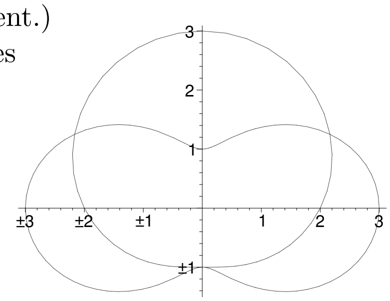
Additional Problems.



Problem J. The first thing is to get a picture of the graph of the polar curve $r^2 = 9\cos 5\theta$.This one is similar to the graph of $r = \cos 2\theta$, done in section 11.3; the result is pictured at right. The area is 5 times the area of one loop, which is described by $-\frac{\pi}{10} \leq \theta \leq \frac{\pi}{10}$. Thus the area is

$$45 \int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} \int_0^{\cos^{1/2}(5\theta)} r dr d\theta = \frac{45}{2} \int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} \cos(5\theta) d\theta = \frac{9}{2} \left[\sin(5\theta)\right]_{-\frac{\pi}{10}}^{\frac{\pi}{10}} = 9.$$

(The answer in the book is 18; I don't see where the factor of 2 went.)



For problem 34, we need a picture of the graphs of the polar curves $r = 2 + \cos 2\theta$ and $r = 2 + \sin\theta$. They are pictured together at right; the first one is the peanut shaped region, and the second is the more circular region. The area is twice the area of the crescent-shaped region at lower right, which ranges from $\theta = -\pi/2$ to the θ that we get from solving $2 + \cos 2\theta = 2 + \sin\theta$. This is given by $\sin\theta = \frac{1}{2}$, i.e., by $\theta = \frac{\pi}{6}$. So

$$A = 2 \int_{-\pi/2}^{\pi/6} \int_{2+\sin\theta}^{2+\cos 2\theta} r dr d\theta$$

which is really too messy to work out.

Problem K. I claim that f must be a constant. This seems pretty obvious, but requires a bit of proof. First, note that $f_x = 0$, so by 1-variable calculus, $f(x, y, z)$ is constant in x, ie can be written as a function $g(y, z)$. Now $g_y = f_y = 0$, so by the same argument g is constant in y, ie can be written as a function $h(z)$. Finally, $h_z = f_z = 0$, so again h is a constant. It follows that f is this same constant.

Problem L. The hypothesis is just a way of writing that $g = f_x$ and $h = f_y$. Now $g_y = f_{xy} = f_{yx}$ (by Clairaut's theorem) which is in turn just h_x. Thus $g_y = h_x$ as required.

Problem M. This is just a particularly succinct form of the chain rule. The less succinct version is $g'(t) = \frac{\partial F}{\partial x} x'(t) + \frac{\partial F}{\partial y} y'(t) + \frac{\partial F}{\partial z} z'(t) = (\nabla F) \cdot \mathbf{r}'(t)$.

Problem N. We showed earlier in the course that the tangent vector to a curve in a level surface is orthogonal to the normal vector to that surface. (This followed from the chain rule: write the curve as $\mathbf{r}(t) = (x(t), y(t), z(t))$, and take $\frac{d}{dt}$ of both sides of the equation $F(x(t), y(t), z(t)) = 0$. By the chain rule, the left-hand side is $\nabla F \cdot \mathbf{r}'(t)$. But ∇F is the normal vector to the surface, and $\mathbf{r}'$ is the tangent vector to the curve.) Since the curve lies in both surfaces, it is orthogonal to ∇F

and also to ∇G, ie orthogonal to both normals.

Problem O. Following the hint, let's look at the product rule(s):

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t), \quad \frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t).$$

From the first one, we have $\int \mathbf{u}(t) \cdot \mathbf{v}'(t)\ dt = \int \frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t))\ dt - \int \mathbf{u}'(t) \cdot \mathbf{v}(t)\ dt$, and using the fundamental theorem of calculus, $\int \frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t))\ dt = \mathbf{u}(t) \cdot \mathbf{v}(t)$. The same argument works for the cross product product rule.