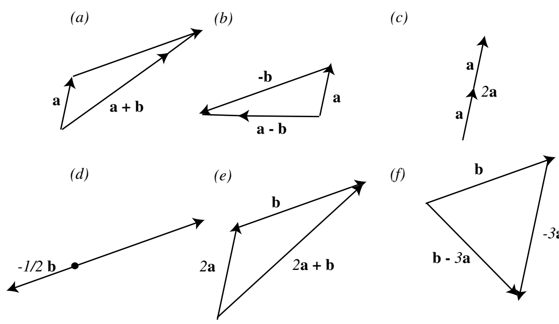
Solutions to Homework 2

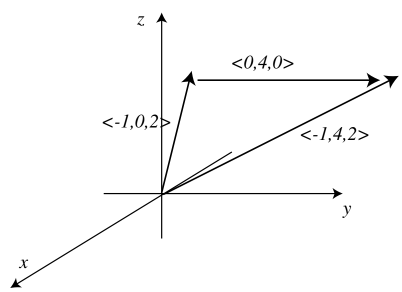
Section 13.2

Problem 4. (a) The vector with initial point P and terminal point R, namely $\vec{PR}$. (b) The vector with initial point R and terminal point S, namely $\vec{RS}$. (c) Think of $-\vec{PS}$ as $\vec{SP}$, then $\vec{QS}-\vec{PS} = \vec{QS}+\vec{SP} = \vec{QP}$. (d) Since $\vec{RS} + \vec{SP} = \vec{RP}$, the sum is $\vec{RP} + \vec{PQ} = \vec{RQ}$.

Problem 6.



Problem 16. $\langle -1, 0, 2\rangle + \langle 0, 4, 0\rangle = \langle -1, 4, 2\rangle$.



Problem 18. $|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$, $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{a} - \mathbf{b} = \mathbf{i} - 8\mathbf{j}$, $2\mathbf{a} = 4\mathbf{i} - 6\mathbf{j}$, and finally $3\mathbf{a} + 4\mathbf{b} = 10\mathbf{i} + 11\mathbf{j}$.

Problem 26. The length of $\mathbf{a} = \langle -2, 4, 2\rangle$ is $\sqrt{4 + 16 + 4} = \sqrt{24}$. To get a vector in the same direction with length 6, we should multiply $\mathbf{a}$ by $6/|\mathbf{a}| = 6/\sqrt{24} = \sqrt{6}/2$. So the answer is $\frac{\sqrt{6}}{2}\mathbf{a} = \langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6}\rangle$. Another way to phrase this is to first find the unit vector in the direction of $\mathbf{a}$, namely $\mathbf{u} = \frac{1}{\sqrt{24}}\mathbf{a}$, and then multiply $\mathbf{u}$ by 6. You get the same result.

Section 13.3

Problem 6. $st + 2s(-t) + 3s(5t) = st - 2st + 15st = 14st$.

Problem 12. $\mathbf{u}$ and $\mathbf{w}$ are perpendicular, so $\mathbf{u}\cdot\mathbf{w} = 0$. You can find $\mathbf{u}\cdot\mathbf{v}$ by trigonometry ($|\mathbf{u}| = 1$, $|\mathbf{v}| = \sqrt{2}/2$, and the angle between them is $\pi/4$, so $\mathbf{u}\cdot\mathbf{v} = 1\sqrt{2}/2\sqrt{2}/2 = 1/2$). Alternatively, observe that $\mathbf{v} = \frac{1}{2}(\mathbf{u}+\mathbf{w})$, so $\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}\mathbf{u}\cdot(\mathbf{u}+\mathbf{w}) = \frac{1}{2}(\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{w}) = \frac{1}{2}(1+0) = \frac{1}{2}$.

Problem 22. We need the sides of the triangle: $\vec{DE} = \langle -2, 3, 2\rangle$, $\vec{EF} = \langle 3, -2, -4\rangle$, and $\vec{FD} = \langle -1, -1, 2\rangle$. Next, we need their lengths: $|\vec{DE}| = \sqrt{4+9+4} = \sqrt{17}$, $|\vec{EF}| = \sqrt{9+4+16} = \sqrt{29}$, and $|\vec{FD}| = \sqrt{1+1+4} = \sqrt{6}$. Let the angles at D, E, and F be $\theta_d, \theta_e, \theta_f$, then their cosines are given by

$$\cos\theta_d = \frac{\vec{DF}\cdot\vec{DE}}{|\vec{DF}||\vec{DE}|} = \frac{-3}{\sqrt{6\cdot 17}} \approx -0.297 \quad \cos\theta_e = \frac{\vec{EF}\cdot\vec{ED}}{|\vec{EF}||\vec{ED}|} \approx 0.9 \quad \cos\theta_f = \frac{\vec{FD}\cdot\vec{F}}{|\vec{FD}||\vec{FE}|} \approx 0.682.$$

So $\theta_d \approx 107°$, $\theta_e \approx 26°$, and $\theta_f \approx 47°$. These add up to 180°, as expected. Look carefully at the calculations above: note that in the first, I took the dot product $\vec{DF}\cdot\vec{DE}$ rather than $\vec{FD}\cdot\vec{DE}$. This is because I wanted two vectors that emanated from the point D, so I reversed the direction of $\vec{FD}$. The same holds true for each of the angles; if you don't do this, you will be finding the complementary angles at each vertex, and they will not add up to 180°.

Problem 24. (a) $\mathbf{u}\cdot\mathbf{v} = -12-108-48 = -168$; not orthogonal. (b) $\mathbf{u}\cdot\mathbf{v} = 2+1+2 = 5$; not orthogonal. (c) $\mathbf{u}\cdot\mathbf{v} = -ab+ba+c\cdot 0 = 0$; orthogonal.

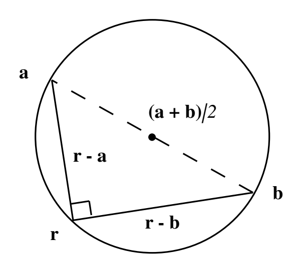
Problem 36. Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 15/5 = 3$. Vector projection is $\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{15}{25}\langle 3, -4\rangle = \frac{3}{5}\langle 3, -4\rangle$.

Problem 44. (a) If $\text{comp}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{b}}\mathbf{a}$ then $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$. If $\mathbf{a}\cdot\mathbf{b} = 0$, then the scalar projections are the same (i.e. 0). If it is not 0, then we can divide out to get the condition $|\mathbf{a}| = |\mathbf{b}|$.
(b) $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$, then $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$. Again, if $\mathbf{a}\cdot\mathbf{b} = 0$, then both projections are the zero vector $\mathbf{0}$. If $\mathbf{a}\cdot\mathbf{b} \neq 0$, then I claim that infact $\mathbf{a} = \mathbf{b}$. To see this, divide by $\mathbf{a}\cdot\mathbf{b}$to get $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2}$. Note that this implies that $\mathbf{a}$ is a positive multiple of $\mathbf{b}$, ie they point in the same direction. Since $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2}$, they have the same length, i.e. $1/|\mathbf{a}| = 1/|\mathbf{b}|$. Hence $\mathbf{a}$ and $\mathbf{b}$ are vectors with the same length pointing in the same direction, so they are equal.

Problem 50. Method 1: write the equation as $(x-a_1)(x-b_1)+(y-a_2)(y-b_2)+(z-a_3)(z-b_3) = 0$, and complete the square to get the sphere

$$\left(x - \frac{a_1+b_1}{2}\right)^2 + \left(y - \frac{a_2+b_2}{2}\right)^2 + \left(z - \frac{a_3+b_3}{2}\right)^2 = \frac{1}{4}((a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2).$$

This is a fair amount of rather unrevealing algebra.
Method 2: The equation $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0$ says that the triangle formed by the points $\mathbf{r}, \mathbf{a}$, and $\mathbf{b}$ is a right triangle with right angle at $\mathbf{r}$, and with hypotenuse $\mathbf{b}-\mathbf{a}$. Elementary geometry says that the set of such points $\mathbf{r}$ is a sphere, with radius 1/2 the length of $\mathbf{b}-\mathbf{a}$, and with center the midpoint of the line from $\mathbf{a}$ to $\mathbf{b}$. Thus the center is (as in the other equation) $\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \langle \frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\rangle$



and the radius is

$$\frac{1}{2}\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}.$$

Method 3. Once you know the answer from the geometry in method 2, you can verify algebraically that it works. Note first that expanding $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b})=0$ gives $\mathbf{r}\cdot\mathbf{r}-\mathbf{r}\cdot(\mathbf{a}+\mathbf{b})=-\mathbf{a}\cdot\mathbf{b}$. The square of the distance from the point $\mathbf{r}$ to the point $\frac{\mathbf{b}+\mathbf{a}}{2}$ is

$$\begin{aligned}\left(\mathbf{r}-\frac{\mathbf{b}+\mathbf{a}}{2}\right)\cdot\left(\mathbf{r}-\frac{\mathbf{b}+\mathbf{a}}{2}\right) &= \mathbf{r}\cdot\mathbf{r}-\mathbf{r}\cdot(\mathbf{a}+\mathbf{b})+\frac{1}{4}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= -\mathbf{a}\cdot\mathbf{b}+\frac{1}{4}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= -\mathbf{a}\cdot\mathbf{b}+\frac{1}{4}(\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b})\\ &= \frac{1}{4}(\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}) = \frac{\mathbf{a}-\mathbf{b}}{2}\cdot\frac{\mathbf{a}-\mathbf{b}}{2}\end{aligned}$$

This says what we expect: the distance from $\mathbf{r}$ to $\frac{\mathbf{b}+\mathbf{a}}{2}$ is constant $=|\frac{\mathbf{a}-\mathbf{b}}{2}|$.

Method 4. Finally, you could view the result from method 2/3 as being just like 'completing the square' in vector form. Here's what I mean by that peculiar sounding statement. The equation of a circle with radius R and center $\mathbf{c}$ is given by $(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c})=R^2$. The left side of that equation can be expanded using the distributive law to give

$$\mathbf{r}\cdot\mathbf{r}-2\mathbf{r}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}=R^2. \tag{1}$$

Now suppose we have an equation of the form $\mathbf{r}\cdot\mathbf{r}-\mathbf{r}\cdot\mathbf{d}=S$ and we want to convert it into an equation of the form (1). As in the usual completing the square, we see that $\mathbf{c}$ should be of the form $\frac{1}{2}\mathbf{d}$, and so we should add $\frac{1}{4}\mathbf{d}\cdot\mathbf{d}$ to both sides to get

$$\begin{gathered}\mathbf{r}\cdot\mathbf{r}-\mathbf{r}\cdot\mathbf{d}+\frac{1}{4}\mathbf{d}\cdot\mathbf{d}=S+\frac{1}{4}\mathbf{d}\cdot\mathbf{d}\quad\text{or}\\ (\mathbf{r}-\frac{1}{2}\mathbf{d})\cdot(\mathbf{r}-\frac{1}{2}\mathbf{d})=S+\frac{1}{4}\mathbf{d}\cdot\mathbf{d}\end{gathered}$$

which is of the desired form (assuming $S+\frac{1}{4}\mathbf{d}\cdot\mathbf{d}$ is positive, so we can write it as R^2.)