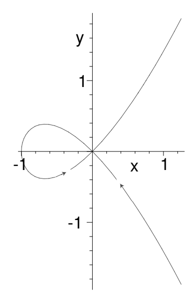
Solutions to Midterm 1 review problems

Chapter 11 (pp. 733-734): 3,4,6,11,12,13, 15,19,22,23,27,31 (and find the angle that the curves make at the intersection points). True-false quiz: 1,2,4,5,6,7,8.
Chapter 13 (pp. 881-882): 1,3,5,6,7,8 (Unless you have a lot of time to kill, use theorem on p. 854 instead of multiplying all this out.), 15,19,20,28,29,31,32(should look familiar from the quiz), 37,38, 40,41,42,43,44,45. True-false quiz: All of them.
Chapter 14 (pp. 918-919): 1,2,5,8,9. True-false quiz: 1,2,3,4,5,6.

Chapter 11

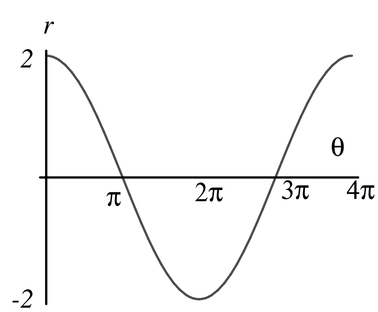
Problem 4. $x = 2\cos\theta$, $y = 1 + \sin\theta$. Plugging into $\sin^2\theta + \cos^2\theta = 1$, we get $x^2/4 + (y-1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1

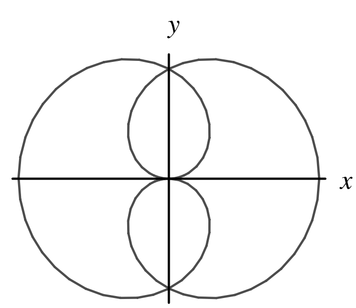
Problem 6. Follow x and y along as t increases, giving



Problem 12. $r = \cos(\theta/2)$.

The curve is symmetric about the pole and both the horizontal and vertical axes.



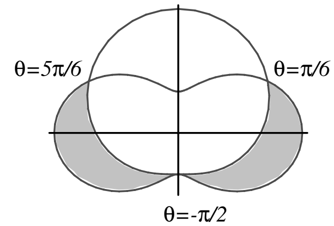


Problem 22. $\frac{dy}{dx} = \frac{-3\sin 3\theta \sin\theta + (3+\cos 3\theta)\cos\theta}{-3\sin 3\theta\cos\theta - (3+\cos 3\theta)\sin\theta}$. When $\theta = \pi/2$, this gives -1.

Problem 31. The points of intersection are when $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$. To find the angle, find the tangent vectors; here is what happens for $\theta = \frac{\pi}{3}$. The curve $r = 2$ is parameterized by $\mathbf{a}(\theta) = \langle 2\cos\theta, 2\sin\theta\rangle$, so its tangent vector is $\mathbf{a}'(\theta) = \langle -2\sin\theta, 2\cos\theta\rangle$. Likewise, the other curve is parameterized by $\mathbf{b}(\theta) = \langle 4\cos^2\theta, 4\cos\theta\sin\theta\rangle = \langle 2\cos 2\theta + 2, 2\sin 2\theta\rangle$ with tangent vector $\mathbf{b}' = \langle -4\sin 2\theta, 4\cos 2\theta\rangle$. At $\theta = \frac{\pi}{3}$, this gives $\mathbf{a}' = \langle -sqrt3, 1\rangle$ and $\mathbf{b}' = \langle -2\sqrt{3}, -2\rangle$. The angle, say ϕ, is given by $\cos\phi = \frac{\mathbf{a}'\cdot\mathbf{b}'}{|\mathbf{a}'||\mathbf{b}'|} = \frac{1}{2}$. So the angle is $\frac{\pi}{3}$. The lower angle is the same, by symmetry.

Problem 34. The curves meet when $2 + \sin\theta = 2 + \cos 2\theta = 3 - 2\sin^2\theta$.

Solving for $\sin\theta$ gives $\sin\theta = -1$ or $\sin\theta = \frac{1}{2}$. Thus $\theta = -\frac{\pi}{2}, \frac{\pi}{6}$ or $\frac{5\pi}{6}$. The curves are plotted at right. The area is given by $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}[(2+\cos 2\theta)^2 - (2+\sin\theta)^2]d\theta = \int_{-\pi/2}^{\pi/6}[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta]d\theta = [2\sin 2\theta + \frac{1}{2}\theta + \frac{1}{8}\sin 4\theta + 4\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta]_{-\pi/2}^{\pi/6} = \frac{51}{16}\sqrt{3}$.



True-False: 1 True, 2 False, 4 False–$\tan\theta = y/x$, but you have to determine what quadrant you are in to distinguish θ from $\theta + \pi$, 5 True, 6 True (they're all circles of radius 2 around the origin), 7 False–the first is the half of the parabola $y = x^2$ where $x \geq 0$, but the second is the whole parabola, 8 True.

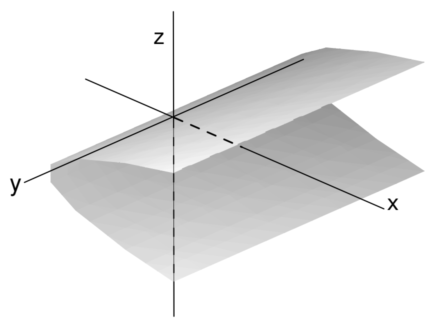
Chapter 13

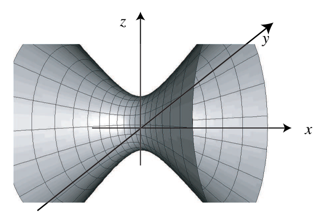
Problem 6. We know that the cross product of two vectors is orthogonal to both. so we calculate $(\mathbf{j}+2\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, which has length $3\sqrt{6}$. The two unit vectors are $\pm\frac{1}{3\sqrt{6}}(7\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

Problem 8. Use the formula from 13.4 (Theorem 8, part 6): $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$. Applying this with $\mathbf{u} = (\mathbf{b}\times\mathbf{c})$, $\mathbf{v} = \mathbf{c}$, and $\mathbf{w} = \mathbf{a}$ gives $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = ((\mathbf{b}\times\mathbf{c})\cdot\mathbf{a})\mathbf{c} - ((\mathbf{b}\times\mathbf{c})\cdot\mathbf{c})\mathbf{a} = ((\mathbf{b}\times\mathbf{c})\cdot\mathbf{a})\mathbf{c}$ since $(\mathbf{b}\times\mathbf{c}) \perp \mathbf{c}$. So $(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})) = (\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\cdot\mathbf{a})\mathbf{c} = [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a})][(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]$. On the other hand, part 5 of Theorem 8 says $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, so the result is $[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a})]^2$ as required.

Problem 20. We need a normal vector, and a point on the plane (which we have: $(1, 2, -2)$). We have one direction lying in the plane, namely the direction vector of the line, say $\mathbf{a} = \langle 2, -1, 3\rangle$. To get another vector, choose a point, say $\langle 0, 3, 1\rangle$, and take $\mathbf{b} = \langle 1, 2, -2\rangle - \langle 0, 3, 1\rangle = \langle 1, -1, -3\rangle$. So a normal vector is $\mathbf{n} = \mathbf{a}\times\mathbf{b} = \langle 6, 9, -1\rangle$, and the equation of the plane is $6(x-1) + 9(y-2) - (z+2) = 0$.

Problem 28. A parabolic cylinder whose trace in the xz plane is the x-axis, and which opens to the right, drawn below.





Problem 32. A hyperboloid of one sheet with axis the x-axis, drawn above.

Problem 38. $r = \sqrt{4+4} = 2\sqrt{2}$, $z = -1$, $\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$. So $\theta = \pi/4$ and the cylindrical coordinates are $(2\sqrt{\ };\pi/2, -1)$. For spherical coordinates, $\rho = \sqrt{4+4+1} = 3$, $\cos\phi = -\frac{1}{3}$, so the spherical coordinates are $(3, \pi/4, \cos^{-1}(-\frac{1}{3})$.

Problem 40. One frustrum of a circular cone with vertex the origin and axis the positive z-axis. Think ice cream cone without the ice cream.

Problem 42. Multiply both sides by r to give $r^2 = r\cos\theta$, or $x^2 + y^2 = x$. (Note that there was already a solution when $r = 0$, so we have not introduced any extraneous solutions in this way.) Moving the x to the left side, and completing the square gives $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, which is a circular cylinder of radius $\frac{1}{2}$, with axis the line $x = \frac{1}{2}, y = 0$.

Problem 44. In cylindrical coordinates this is $r^2 = 4$; in spherical coordinates we have the relation that $r^2 + z^2 = \rho^2$, and $z = \rho\cos\phi$. So the equation is $\rho^2 - \rho^2\cos^2\phi = 4$, or $\rho^2\sin^2\phi = 4$. Since ρ and $\sin\phi$ are positive, this is equivalent to $\rho\sin\phi = 2$.

True-False: 1 True, 2 False, 3, True, 4 True, 5 True, 6 True, 7 False, 8 True–use part 6 of Theorem 8 to compute both sides, 9 True, 10 True (distributive law and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$), 11 False–the length depends on the angle between the vectors, 12 False–it's a plane, 13 False–it's a cylinder (because they declared it to be a set of points in 3-space), 14 Very False–the dot product of vectors is a scalar, not a vector.

Chapter 14

Problem 2. (a) We need $t \leq 2$, $t \neq 0$, and $t > -1$ so that the three coordinate functions are all defined. So the domain is $(0, 2] \cup (-1, 0)$.

(b) Taking the limit in each component gives $\langle\sqrt{2}, \lim_{t\to 0} \frac{e^t - 1}{t}, \ln(1)\rangle = \langle\sqrt{2}, \lim_{t\to 0} \frac{e^t}{1}, 0\rangle = \langle\sqrt{2}, 1, 0\rangle$ where I used L'Hôpital's rule in the second step.
(c) $\mathbf{r}'(t) = \langle -\frac{1}{2\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1}\rangle$.

Problem 8. $\mathbf{r}'(t) = \langle 3t^{1/2}, -2\sin 2t, 2\cos 2t\rangle$, so $|\mathbf{r}'(t)| = \sqrt{9t+4}$. $L = \int_0^1 \sqrt{9t+4}\,dt = \left[\frac{2}{27}(9t+4)^{3/2}\right]_0^1 = \frac{2}{27}(13^{3/2} - 8)$.

Problem 18. Integrate once to get $\mathbf{v}(t) = 3t^2\mathbf{i} + 4t^3\mathbf{j} - 3t^2\mathbf{k} + \mathbf{c}_1$. Plug in at $t = 0$ to get $\mathbf{c}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k}$, so $\mathbf{v}(t) = 3t^2\mathbf{i} + 4t^3\mathbf{j} - 3t^2\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k}$. Integrate again to get $\mathbf{r}(t) = (t^3 + t)\mathbf{i} + (t^4 - t)\mathbf{j} + (t - t^3)\mathbf{k} + \mathbf{c}_2$. Since the starting point is the origin, $\mathbf{c}_2 = \mathbf{0}$, so $\mathbf{r}(t) = (t^3 + t)\mathbf{i} + (t^4 - t)\mathbf{j} + (t - t^3)\mathbf{k}$.

True-false: 1 True, 2 True, 3 False, since $\mathbf{r}'(0) = \mathbf{0}$, 4 True, 5 False–you need to use the product rule, 6 False–consider the curve $\mathbf{r}(t) - \langle\cos t, \sin t\rangle$ Then $|\mathbf{r}(t)| = 1$, so $\frac{d}{dt}|\mathbf{r}(t)| = 0$. But $|\mathbf{r}'(t)| = 1$.