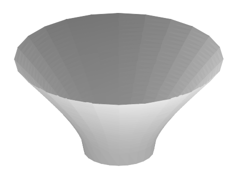
Solutions to review problems for Midterm 2

March 29, 2005

1 Solutions to problems from the book

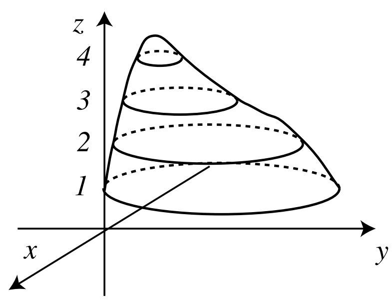
Problem 2. $D = \{(x, y, z) \mid z \geq x^2 + y^2\}$, which is the set of points above the paraboloid $z = x^2 + y^2$.

Problem 4. $z = \sqrt{x^2 + y^2 - 1}$, so $z \geq 0$ and $x^2 + y^2 - z^2 = 1$.



So the graph is the upper half of a hyperboloid of one sheet.

Problem 8.



Problem 10. as $(x, y) \to (0, 0)$ Along the line $x = y$, the value of $\frac{2xy}{x^2+2y^2}$ is $\frac{2x^2}{3x^2}$, so the limit as $(x, y) \to (0, 0)$ along this line is $\frac{2}{3}$. In contrast, along the line $x = -y$, $\frac{2xy}{x^2+2y^2}$ is $\frac{-2x^2}{3x^2} = -frac23$, so the limit does not exist.

Problem 16. $\frac{\partial w}{\partial x} = \frac{1}{y-z}$, $\frac{\partial w}{\partial y} = \frac{-x}{(y-z)^2}$, and $\frac{\partial w}{\partial z} = \frac{x}{(y-z)^2}$.

Problem 22. $v_{rr} = 0$, $v_{rs} = -\sin(s + 2t)$, $v_{rt} = -2\sin(s + 2t)$, $v_{ss} = -r\cos(s + 2t)$, $v_{st} = -2r\cos(s + 2t)$, and $v_{tt} = -4r\cos(s + 2t)$. The others are determined by these, by Clairaut's theorem.

Problem 24. $\frac{\partial^2 \rho}{\partial x^2} = \frac{\partial}{\partial x}(x(x^2 + y^2 + z^2)^{-1/2}) = (x^2 + y^2 + z^2)^{-1/2} - x^2(x^2 + y^2 + z^2)^{-3/2} = \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$. The y and z derivatives are similar; adding them up together gives

$$\frac{(y^2 + z^2) + (x^2 + z^2) + (x^2 + y^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{\rho}.$$

Problem 28. Write $F(x,y,z) = xy+yz+zx-3$, so $F_x = y+z$, $F_y = x+z$, and $F_z = x+y$. Thus the normal vector to the plane is $\langle 2,2,2\rangle$, and the equation of the plane is $2(x-1)+2(y-1)+2(z-1) = 0$.

Problem 36. $\frac{\partial z}{\partial x} = -y\sin xy - y\sin x$, and $\frac{\partial z}{\partial y} = -x\sin xy + \cos x$. So

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (-y\sin xy - y\sin x)2u + (-x\sin xy + \cos x)(1).$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (-y\sin xy - y\sin x)(1) + (-x\sin xy + \cos x)(-2v).$$

Problem 42. Use implicit differentiation. For the first, take the x-derivative of both sides: $4yz^3\frac{\partial z}{\partial x} + 2xz^3 + 3x^2z^2\frac{\partial z}{\partial x} = yze^{xyz} + xy\frac{\partial z}{\partial x}e^{xyz}$. Collecting terms involving $\frac{\partial z}{\partial x}$ gives

$$(4yz^3 + 3x^2z^2 - xye^{xyz})\frac{\partial z}{\partial x} = yze^{xyz} - 2xz^3.$$

So

$$\frac{\partial z}{\partial x} = \frac{yze^{xyz} - 2xz^3}{4yz^3 + 3x^2z^2 - xye^{xyz}}.$$

A similar gruesome calculation gives

$$\frac{\partial z}{\partial x} = \frac{xze^{xyz} - z^4}{4yz^3 + 3x^2z^2 - xye^{xyz}}$$

Problem 44. (a) When the direction vector $\mathbf{u}$ points in the same direction as the gradient ∇f. (b) When the direction vector $\mathbf{u}$ points in the opposite direction to the gradient. (c) When $\mathbf{u}$ is perpendicular to the gradient. (d) When $\cos(\theta) = \frac{1}{2}$, ie when the angle between $\mathbf{u}$ and ∇f is $\frac{\pi}{3}$.

Problem 46. $\nabla f = \langle 2xy + \sqrt{1+z}, x^2, -\frac{x}{2\sqrt{1+z}}\rangle = \langle 6, 1, \frac{1}{4}\rangle$ at $(1,2,3)$. The unit vector in the given direction is $\frac{1}{3}\langle 2,1,-2\rangle$. Taking the dot product gives $D_{\mathbf{u}}f(1,2,3) = \frac{25}{6}$.

2 Solutions to additional problems

Problem A. Write $\alpha(t) = (x(t), y(t)$, and suppose we are at the point $(x_0, y_0) = \alpha(t_0)$ on the graph. Note that we have the equation $F(x(t), y(t)) = 0$. Differentiating with respect to t, and applying the chain rule gives $F_x(x_0,y_0)x'(t_0) + F_y(x_0,y_0)y'(t_0) = 0$. In other words, the vectors $\langle < F_x(x_0,y_0), F_y(x_0,y_0)\rangle$ and $\langle x'(t_0), y'(t_0)\rangle$ are orthogonal, so slopes of lines that these vectors span are inverse reciprocals. (I.e one is -1/slope of the other.) Now the slope of the tangent line which is parallel to $\alpha'(t_0)$, is given by $y'(t_0)/x'(t_0)$, and the slope of the line parallel to

$\langle < F_x(x_0, y_0), F_y(x_0, y_0)\rangle$ is $F_y(x_0, y_0)/F_x(x_0, y_0)$. So $y'(t_0)/x'(t_0) = -F_x(x_0, y_0)/F_y(x_0, y_0)$.

Problem B. $f(0,0) = a_0$, $f_x(0,0) = a$, and $f_y(0,0) = b$. The linearization is therefore given by $a_0 + ax + by$. My guess is that the linearization at $(0,0)$ of a polynomial of high degree is just the constant and the linear terms.

Problem C. Why not just let $g(x) = F(x, 2) = 8\sin(2x) + e^2$?

Problem D. A suggested in the hint, there's no θ in this equation. That means that the surface looks the same in every direction (as measured by θ) or in other words that it has circular symmetry around the z-axis.

Problem E. There is no such function. For the directional derivative is given by $\nabla f(x_0, y_0) \cdot \mathbf{u}$, and if this is negative for one value of $\mathbf{u}$, it will be positive for the opposite direction given by $-\mathbf{u}$.

Problem F. We can discuss this at the review.

Problem G. There are many possible solutions to this problem; here's one. Start with $f(x, y) = \sqrt{x^2 + y^2}$, whose contour line at height k is the circle of radius k. We want the same contours for all whole numbers k, but different contours for other k. For instance $g(x, y) = \cos(2\pi\sqrt{x^2 + y^2})\sqrt{x^2 + y^2}$ will work, since $\cos(2\pi\sqrt{x^2 + y^2})$ is 1 whenever $\sqrt{x^2 + y^2}$ is a whole number, and less than 1 otherwise.

Problem H. The main observation is that any function written as a sum of other functions, say $f(x, y) = g(x, y) + h(x, y)$, all of the derivatives of f are sums of the corresponding derivatives of g and h. In particular, $f_{xy} = g_{xy} + h_{xy}$ and $f_{yx} = g_{yx} + h_{yx}$. So if Clairaut's theorem holds for g and h, it must hold for f. Similarly, if Clairaut's theorem holds for a function f, it holds for cf where c is any constant. Since any polynomial is a sum of terms of the form $cx^m y^n$, Clairaut's theorem for polynomials will follow once we've proven it for terms of the form $x^m y^n$. We do that by direct calculation: $\frac{\partial^2}{\partial x \partial y}(x^m y^n) = mnx^{m-1}y^{n-1} = \frac{\partial^2}{\partial y \partial x}(x^m y^n)$.