NOTE: There is a lot of material to review for the final exam. I would suggest a careful going over of the midterm and quizzes, especially the problems you didn't get, and likewise a look over the homeworks. Here are some suggestions for problems from the last few sections, followed by some problems including material from earlier in the course. I haven't included any problems such as finding an inverse, or getting the complete solution to some set of equations, as you can find plenty of those in the early sections of the book.
§6.1: 7, 19, 23, 29.
§6.2: 7, 13, 25.
§6.3: 7, 9, 17, 19, 24 (we did this in class, but try without looking at your notes), 26. 27 (a better hint: think about lengths of vectors), 29, 33, 34, 35, 38.
§8.1: 11, 13, 15.
§8.3: 5, 7, 13, 17.

1. Let $\vec{v}_1 = [1, 2]$. Find a vector $\vec{v}_2 \in \mathbb{R}^2$ which makes an angle of $\pi/3$ with $\vec{v}_1$. Here's a harder version: Let $\vec{w}_1 = [1, 1, 1]$. Find a vector $\vec{w}_2 \in \mathbb{R}^3$ which makes an angle of $\pi/3$ with $\vec{w}_1$.

2. Let A be the matrix
$$\begin{bmatrix} 1 & -2 & 1 & 0 & 2 \\ 2 & -4 & 2 & 1 & 7 \\ 1 & -2 & 1 & -1 & -1 \end{bmatrix}$$
   (a) Find an orthogonal basis for the null-space of A.
   (b) Find a basis for the orthogonal complement of the null-space of A.

3. Is $\{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{for some } s, t \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$? Briefly justify your answer.

4. (a) Give the definition of the kernel of a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$.
   (b) Give an example of a linear function T with $\ker(T) = \{\vec{0}\}$.
   (c) Give an example of a linear function T with $\ker(T) \neq \{\vec{0}\}$.
   (d) Prove: Let T be a linear function. If $\ker(T) \neq \{\vec{0}\}$ then T is not one-to-one.
   (e) Give an example of a linear function $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is not onto.

5. Prove that if A is similar to B, then A^T is similar to B^T.

6. Find the matrix for the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ that is given as the composition of these three transformations (performed in the order given): T_1 = rotation of $\pi/2$ in the yz plane, T_2 = rotation of $\pi/2$ in the xz plane, T_3 = rotation of $\pi/2$ in the xy plane. (So you do T_1 first, then T_2, then T_3.)

7. Let A be an *anti*diagonal matrix, *i.e.* a matrix of the form:

$$\begin{bmatrix} 0 & \dots & \dots & 0 & a_1 \\ 0 & \dots & \dots & a_2 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & a_{n-1} & 0 & \dots & 0 \\ a_n & 0 & \dots & \dots & 0 \end{bmatrix}$$

(a) Prove that $\det(A) = \pm a_1 \cdot a_2 \cdots a_n$.

(b) Figure out the sign (as a function of n).

8. Let A be an $n \times n$ matrix and that $\vec{v}_1, \dots, \vec{v}_n$ are vectors such that $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is independent. Show that A is invertible. (Hint: first show that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is independent.)

9. Suppose that $\vec{u}$ and $\vec{v}$ are parallel (non-zero) vectors in $\mathbb{R}^n$, and that $\vec{w}$ is another non-zero vector in $\mathbb{R}^n$ that makes an angle θ with $\vec{u}$. Show (using dot products, not geometry) that the angle between $\vec{v}$ and $\vec{w}$ is either θ or $\pi - \theta$, and explain when the two alternatives occur.

10. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ be linear. Show (from the definition of linearity) that the composition $g \circ f$ is linear.

11. Let A be an $n \times n$ matrix with *distinct* eigenvalues $\lambda_1, \dots, \lambda_n$.
(a) What is $\det(A)$?
(b) Suppose that each eigenvalue satisfies $|\lambda_i| < 1$. What can you say about the volume of the parallelepiped spanned by the columns of A?
(c) Suppose that each eigenvalue satisfies $|\lambda_i| < 1$. What can you say about the behavior of A^k for k large (i.e. as $k \to \infty$).

12. Let $\vec{u} \in \mathbb{R}^2$ be a non-zero vector, and let $\vec{v}$ be any vector in $\mathbb{R}^2$.
(a) Show that there is a linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(\vec{u}) = \vec{v}$.
(b) If in addition $\vec{v} \neq \vec{0}$, show there is an f which is a one–to–one function.

13. Suppose that A is an $n \times n$ matrix with $A^2 = A$, and that λ is an eigenvalue of A. (a) Show that $\lambda = 0$ or $\lambda = 1$. [Hint: Apply A to both sides of the equation $Ax = \lambda x$.] (b) If 0 is **not** an eigenvalue, show that $A = I$. (c) Give an example where $A \neq I$.

14. Let B be an $n \times n$ matrix with $\det(B) = 0$. Show that $\text{rank}(B) \leq n - 1$.

15. Let X be the following set of vectors in $\mathbb{R}^3$

$$X = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(a) Show that X is linearly dependent. (b) Find (with justification) a basis for $\mathbb{R}^3$ consisting of vectors from X.

16. Let P be the plane throught the origin in $\mathbb{R}^3$ perpendicular to the vector $\vec{n} = (-1, 1, 1)$.
(a) Find a basis $\{\vec{u}, \vec{v}\}$ for P.
(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by projection onto P. What is the matrix of T, and what are its eigenvalues?

17. Prove the 'spectral theorem' for 2×2 matrices: Let A be a symmetric 2×2 matrix. Then there is an orthogonal basis for $\mathbb{R}^2$ consisting of eigenvectors of A. (Hint: First try to show that the the characteristic polynomial of A has real roots. You may have to treat separately the case when there is a double root of the polynomial.)

18. Let A be the matrix:

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

(a) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors for A.
(b) Find a matrix C with $C^{-1}AC =$ a diagonal matrix D.
(c) Find D^4.
(d) Find A^4.

19. Your mean calculus teacher has asked you to figure out if $f(x_1, x_2, x_3) = [x_1, x_2, x_3]\ A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where A is the symmetric matrix $\begin{bmatrix} 444 & 26 & 39 \\ 26 & -15 & 12 \\ 39 & 12 & -25 \end{bmatrix}$ has a local max, min, or saddle point at $(0, 0, 0)$. You are struggling with this problem, when your fairy godmother appears, and tells you the following information about A: there is at least one *negative* eigenvalue for A, and $\det(A) > 0$. Can you answer the question now?