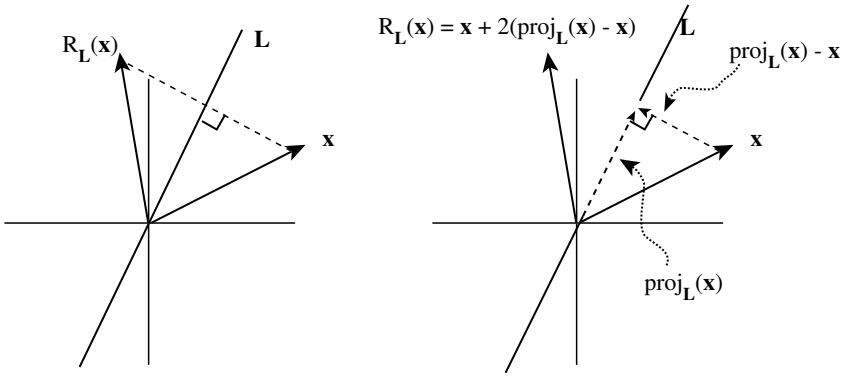
In class work: reflections

Let L be a line through the origin in the plane $\mathbb{R}^2$. We want to discuss the linear transformation $R_L : \mathbb{R}^2 \to \mathbb{R}^2$ gotten by reflection in L. Here's the picture to keep in mind:



The reflection of the vector $\vec{x}$, or $R_L(\vec{x})$ is gotten by dropping a perpendicular from $\vec{x}$ to L, and then extending it to the other side, as in the first figure above.

This next paragraph is just for your information; I don't expect we are going to cover this. Here's how to find a formula for the reflection. This is shown in the second half of the figure. The perpendicular that we dropped lands exactly at $\text{proj}_L(\vec{x})$, so the vector from $\vec{x}$ to that projection is $\text{proj}_L(\vec{x}) - \vec{x}$. To get the reflection, we add another copy of that vector (to keep going to the other side of L). Hence the reflection is given by

$$R_L(\vec{x}) = \vec{x} + (\text{proj}_L(\vec{x}) - \vec{x}) + (\text{proj}_L(\vec{x}) - \vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}.$$

If L is the line spanned by the vector $\vec{v}$, then $\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$, so we get the formula $R_L(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} - \vec{x}$.

We will only use the reflections in the x-axis, the y-axis, and the line $x = y$, and will denote each of these by R_x, R_y, and R_{xy}. Each of these is represented by a matrix, say A_x, A_y, or A_{xy}.

Now: team up with one or two of your neighbors, and do the following exercises. First, pick two of the reflections (eg pick R_y and R_{xy}) that you are going to work with. Since I don't know which you picked, I'll refer to one as R and the other as R'. **1.** Find the corresponding matrices A and A' (so if your R is R_x, you should find A_x).

2. Find the composition of functions $R \circ R'$ (ie find, geometrically, $R(R'(\vec{x}))$ for $\vec{x} \in \mathbb{R}^2$.)

3. Find the composition of functions $R' \circ R$.

4. Find the compositions $R \circ R$ and $R' \circ R'$. Why did you get the results you got? **5.** Find the matrix products AA', $A'A$, A^2, and $(A')^2$. Are your results consistent with your answers to **2**, **3**, and **4**?

Now draw two lines L and L', or use the figure below, and let $R = R_L$ and $R' = R_{L'}$.

6. Guess a geometric description for $R \circ R'$ and $R' \circ R$.