Math 21a, Fall 2004. **Review sheet for exam 2, Monday, Nov. 22**

The second midterm will cover material from sections 2.1-2.3 (plus the basics about rotations and reflections in the plane, as discussed in class), 3.4 (material about linear transformations), 4.1-4.4, 5.1 and 5.2. You are still responsible for what you learned in chapter 1 about linear equations and matrices. As on the first midterm, there will be both computational and conceptual problems. (I don't think these are really such different things!)

In studying, you may find the "summary" at the end of each section to be helpful. Here is a list of the basic ideas that we've covered:

1. Linear independence/dependence and related notions (linear relations, called dependence relations in the book, how a proof that some vectors are independent should start and end.)
2. Spanning set for a subspace.
3. Basis of a subspace, dimension, and basic facts (k independent vectors in a k-dimensional subspace automatically span, etc.)
4. How to find bases for null space, column space, row space of a matrix, and kernel and image of a linear transformation.
5. Rank and nullity of a matrix; rank + nullity theorem (called *Rank Equation* in the book).
6. Linear Transformations; matrix of a linear transformation, T is determined by its effect on a basis.
7. One-to-one and onto functions (in general and what these concepts mean for linear transformations); relation to rank and nullity of a matrix. Compositions and inverses of linear transformations.
8. Rotations of the plane, projection and reflection in a line in the plane.
9. Cross products of vectors in $\mathbb{R}^3$; expression as a determinant, relation to volume of parallelepiped and area of parallelogram in $\mathbb{R}^3$. Algebraic properties ($\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, etc.) Geometric significance of $\vec{a}\times\vec{b}$.
10. Basic properties of determinants: behavior under row operations, relation to invertibility and linear independence of rows/columns, expansion along minors, computing by reduction to echelon form. Determinants of special matrices (diagonal, triangular, etc.).
11. Volume and determinants; $\det(A)$ as the 'stretching factor' for volume. (You **don't** have to know the formula $\sqrt{\det(A^TA)}$ for the area of an n-box in $\mathbb{R}^m$, but I do expect you to be able to compute the area of a parallelogram in $\mathbb{R}^3$.)
12. Eigenvalues and eigenvectors; basic definitions and how to find them. Finding a basis for an eigenspace. How to pronounce the word 'eigenvector'.
13. Diagonalization of matrices; how to carry it out in practice. Diagonalization of special matrices such as reflections/projections. Similar matrices.

I would suggest that you do some problems from the book as part of your studying. Here are some suggestions, although probably you don't want to do them all. First, each section has True/False questions; these are very useful as review questions. In addition, I suggest the following problems. When you do the problems in the earlier sections, use the knowledge you've gained in the later sections.
§2.1 11, 24, 25, 31, 34, 35, 37.
§2.2 4, 5, 6, 7, 21, 22, 23

§2.3 1, 4, 7, 10, 15, 18, 21-28, 34 (refer to the book or the reading on functions for the inverse image, which I called the preimage).
§2.4 3, 4, 5, 6. You might also try 6 using diagonalization, as in §5.2, p. 309.
§3.4 36, 48.
§4.1 7, 13, 17, 23, 27, 32, 33, 35, 39, 47, 51, 55,
§4.2 9, 10, 28, 29, 33, 34.
§4.3 8, 9, 11, 22, 23 (for a 2×2 matrix)
§4.4 1, 7, 13, 20, 21, 24, 25.
§5.1 5, 7, 9, 11, 15, 19, 21, 31, 35, 37, 41, 44
§5.2 3, 7, 9, 12, 17 (hint: what is the constant term of the characteristic polynomial?), 22, 24.

Some additional problems.
Here are a few more theoretical problems.

A. Let V and W be subspaces of $\mathbb{R}^n$. Show that $V \cap W$ is a subspace of $\mathbb{R}^n$.

B. (i) Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}$.

(ii) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find the matrix for linear transformations $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $U : \mathbb{R}^3 \to \mathbb{R}^3$ such that $S(\vec{e}_1) = \vec{v}_1$ (etc.) and $U(\vec{v}_1) = \vec{e}_1$, etc.

(iii) Find the matrix for a linear transformation $V : \mathbb{R}^3 \to \mathbb{R}^4$ with $V(\vec{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $V(\vec{v}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, and $V(\vec{v}_3) = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. (You're supposed to use (i) and (ii) to solve (iii).)

C. Suppose that P is a plane in $\mathbb{R}^3$ with basis $\vec{v}_1, \vec{v}_2$ and that $\vec{v}_3$ is a nonzero vector perpendicular to P. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent (and hence a basis for $\mathbb{R}^3$). Suggestion: after you write down the usual start of a proof of linear independence, think about $\vec{v}_3 \cdot \vec{v}_3$.

D. Give examples of 4×5 matrices, if possible, with ranks 0, 1, 2, 3, 4, and 5.

E. If the rank of a 5×3 matrix A is 3, what is rref(A)? If the rank of a 4×4 matrix B is 2, what are the possibilities for rref(B)?

F. Show that if A and B are 2×2 matrices representing rotations by angles α and β respectively, then A and B commute (i.e., $AB = BA$). Is this true if instead A and B represent reflections?

G. What is the volume of the region of $\mathbb{R}^3$ enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$? **Hint:** we know that the volume of the unit sphere is $\frac{4}{3}\pi$. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that takes the unit sphere to the ellipsoid.