A few review problems about subspaces and bases

These are some practice problems to help you get ready for the makeup quiz on subspaces and bases. You should review chapter 1, especially section 1.6.

(0) You should know the definitions of the various concepts: subspace, basis, linear combination, span, null space, etc.

A. There are some problems in section 3.2 about subspaces/bases; where they say 'the vector space V', you should interpret it as saying $\mathbb{R}^n$. I suggest 3.2: 9, 10, 27, 29, 30, 31, 32 (In problems 31 and 32, take V to be a subspace of $\mathbb{R}^n$. The verb 'generate' is synonymous with 'span'.)

B. Suppose that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for W, and that $\vec{w}$ is a vector of W that is different from $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Show that $\{\vec{w}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is *not* a basis for W.

C. Suppose that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for W. Which of the following sets of vectors are a basis for W?

1. $B_1 = \{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_4, \vec{v}_4 + \vec{v}_1\}$.
2. $B_2 = \{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_3 - \vec{v}_4, \vec{v}_4 - \vec{v}_1\}$.
3. $B_3 = \{17\vec{v}_1 + 43\vec{v}_2 - 22\vec{v}_3 + 6\vec{v}_4, 19\vec{v}_1 - 13\vec{v}_3 + \vec{v}_4, 23\vec{v}_4 - 184\vec{v}_1\}$.

D. Let A be an $m \times n$ matrix, and let V be a subspace of $\mathbb{R}^m$.

1. Let $W = \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{v} \text{ for some } \vec{v} \in V\}$. Show, from the definition of subspace, that W is a subspace.
2. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for V, then show that $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ spans W.
3. Is $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ a basis for W? Give examples where it is a basis, and examples where it is not a basis. What property or properties of A might ensure that it is a basis?