1 Set theory: basic concepts, the meanings of some words

We already started dealing with sets:

- The meaning of a sentence is the set of situations in which it is true.
- Entailment: sentence A entails sentence B if the set of situations in which B is true is contained within the set of situations in which A is true.

(1) a. Fido is a dog. b. Fido is a poodle.
(2) a. John has one good leg. b. John has two good legs.

(3) A set is a collection of objects.

(4) An object is an element of a set A if that object is a member of the collection A.
Notation: “∈” reads as “is an element of” or “belongs to”.

(5) A set A is a subset of a set B if all the elements of A are also in B.
Notation: “⊆” reads as “is a subset of”.

QUESTION 1. In the definition of entailment above, does the phrase “is contained within” correspond to “is an element of” or to “is a subset of”?

We can treat meanings of some nouns, verbs, and adjectives as sets:

- Nouns: denotations of nouns may vary from world to world. A noun denotes a set.
- Adjectives: their denotations also vary. An adjective like red or wooden or blond is a set.
  - red is the set of all the red things in the world or situation,
  - apple is the set of all the apples in the world or situation

If in situation M Betty and Connor smoke and Ann doesn’t, we write

\[
[[\text{smoke}]]^M = [[\text{smoker}]]^M = \{\text{Betty, Connor}\} = \{x: x \text{ smokes in } M\}
\]

And if Connor and Ann are blond, while Betty is a brunette,

\[
[[\text{blond}]]^M = \{\text{Ann, Connor}\} = \{x: x \text{ is blond in } M\}
\]
QUESTION 2. In the example (1) above, what is the relationship between the meanings of the words dog and poodle? What about the meanings of the words dog and Fido?

(6) The **intersection** of two sets $A$ and $B$ ($A \cap B$) is the set containing all and only the objects that are elements of both $A$ and $B$.

(7) The **union** of two sets $A$ and $B$ ($A \cup B$) is the set containing all and only the objects that are elements of $A$, of $B$, or of both $A$ and $B$.

(8) The **complement** of a set $A$, written $A'$, is the set containing all the individuals in the discourse except for the elements of $A$.

We can combine meanings of some set-denoting words by intersecting the sets they denote. For instance, with the $M$ as described above, we can say that

$\llbracket \text{blond smoker} \rrbracket^M_M = \llbracket \text{blond} \rrbracket^M_M \cap \llbracket \text{smoker} \rrbracket^M_M = \{ \text{Ann, Connor} \} \cap \{ \text{Betty, Connor} \} = \{ \text{Connor} \} = \{ x : x \text{ is a blond smoker in } M \}$

(9) Situation $M$:

$U = \{ \text{Ann, Betty, Connor, apple1, apple2, apple3, cherry, table, chair} \}$

(10) $\llbracket \text{red} \rrbracket^M_M = \{ \text{apple1, apple3, cherry, chair} \}$

$\llbracket \text{wooden} \rrbracket^M_M = \{ \text{table, chair} \}$

$\llbracket \text{green} \rrbracket^M_M = \{ \text{apple2, table} \}$

QUESTION 3: Give the denotation of red apple (in both variants) in world w100. Do the same for red wooden thing.

(11) The **power set** of a set $A$ (written as $\wp(A)$) is the set whose members are all the subsets of $A$.

QUESTION 4: Given the sets under (12) and assuming that the universe of the discourse is $\cup\{ A, B, C, D, E, F, G \}$, list the members of the following sets:

(12) $A = \{ 1, 2, 3, 4 \} \quad E = \{ \{ 1 \}, 2, \{ a, 1 \} \}$
\[
B = \{a, b, c, d, e, f\} \quad F = \{1, c, d\} \\
C = \{1, 2\} \quad G = \{d, e, 2, 3\} \\
D = \{1, 3, 4, a, b\}
\]

(13) 
\begin{align*}
\text{a. } & C-D = \\
\text{b. } & A \cap F = \\
\text{c. } & A \cap B = \\
\text{d. } & C \cap F' = \\
\text{e. } & E \cap C = \\
\text{f. } & (C \cup D) - (C \cup D) = \\
\text{g. } & F \cap C = \\
\text{h. } & G \cap C = \\
\text{i. } & A \cap E = \\
\text{j. } & (E \cup B) \cup D = 
\end{align*}

2 Relations and functions

Ordered Pairs and Cartesian Product:

(14) Ordered pair/n-tuple: a set with n-elements where order matters.
\[
<a,b>
\]

Relations: a relation is a set of pairs (or, more generally, of n-tuples).
E.g., "mother of", "to be sitting to the right of".

Relation in \(A\). 
E.g. "advisor of" in the set of people.

Relation from \(A\) to \(B\). 
E.g. "advisor of" from the set of professors to the set of students

(15) A relation from \(A\) to \(B\) is a set of pairs whose first element is from \(A\) and the second element is from \(B\). A relation in \(A\) is a set of pairs whose first and second elements are from \(A\).

(16) Inverse of a relation: \(R^{-1}\)
\[
R^{-1} =_{df} \{<b,a> \mid <a,b> \in R\}
\]

QUESTION 5: Give the denotation of \text{kiss} in situation \(M\). Do the same for \text{assign}.


\begin{align*}
\text{(18) } & \llbracket \text{kiss} \rrbracket^M = \\
\text{(19) } & \llbracket \text{assign} \rrbracket^M = 
\end{align*}
QUESTION 6: What is the relationship between a relation denoted by a transitive verb, and its passive form? For instance, the relationship between kiss and be kissed by?

(20) A relation $R$ from $A$ to $B$ is a function from $A$ to $B$ ($F:A \rightarrow B$) iff:
   a. Every member of $A$ appears at least once as first member of a pair.
   b. Every member of $A$ appears at most once as first member of a pair.

(21) Argument and value: $F(a) = b$

The characteristic function of a set:

(22) a. Let $A$ be a set. $\text{char}_A$, the characteristic function of $A$, is the function $F$ such that, for any $x \in A, F(x)=1$, and for any $x \notin A, F(x)=0$.
   b. Let $F$ be a function whose pairs have second values from the set $\{0,1\}$. Then, $\text{char}_F$, the set characterized by $F$, is $\{x \in D \mid F(x)=1\}$

3 Model-theoretic semantics

Central question: what kinds of model structures are appropriate for interpreting natural language?
   Assumption: what works for one language, should apply to all the others.
   So, part of our task: a semantic theory applicable to every language,
   showing the universal properties of interpretation
   and mapping the range of semantic variation

Syntax tells us what strings (of words) are sentences of a language
Semantics tells us about the meanings of different expressions in a language
   Interpretation: a way of assigning meanings to expressions

Interpretation, at a minimum
What is needed to interpret a language? Starting with a minimum:

A model:
   • universe of things/individuals – domain of discourse, or ‘entities’ (called $A$ in the book)
     $D_e = \{ \text{entities} \}$
   • truth values
     $D_t = \{0, 1\}$
Then, we can have meanings for proper names, and for sentences. We can also build meanings for other expressions:

- predicates (VPs), nouns, and (some) adjectives sets of individual entities
  OR functions from $D_e$ to $\{1,0\}$ (characteristic functions of sets)
- two-place predicates (transitive verbs) are sets of ordered pairs of entities
  OR functions from $D_e$ to other functions, from $D_e$ to $\{1,0\}$
- three-place predicates (ditransitive verbs) are sets of ordered triples of entities
  etc.

**Interpretation function $I$ (called $F_M$ in the book):**

- a function from expressions in the language to things they stand for in the model

Our way of saying $I(Bill)$ or $I(b) = \uparrow$ (a certain entity in the model, the guy named Bill) is

(23) $\llbracket Bill \rrbracket = \uparrow$

We will often use shorthand, and use expressions in Predicate Logic to stand for meanings of expressions in English (we can, because going from PrL to the model is so straightforward), e.g.

(24) $\llbracket Bill \rrbracket = b$

**Compositionality:**

- For complex expressions, we will need a *recipe* for determining what $I$ (or $\llbracket \ldots \rrbracket$) will assign, *based on* what $I$ assign to parts of the expression, and the way the expression is put together

Let’s try now to give a semantics for PrL.

4 **Semantics of Predicate Logic**

*Takes one*, starting with the terms:

(25) a. If $\alpha$ is a constant (excluding syncategorematically treated symbols), then $\llbracket \alpha \rrbracket^M$ is specified by a function $I$ (in the model $M$) that assigns set-theoretical objects to each constant (this is like saying that the semantic value of those constants is fixed in the Lexicon).

b. If $\alpha$ is a variable, then ??????

(26) a. If $P$ is a $n$-ary predicate and $t_1 \ldots t_n$ are all terms, then, for any $M$, $\llbracket P(t_1 \ldots t_n) \rrbracket^M = 1$ iff $< \llbracket t_1 \rrbracket^M, \ldots, \llbracket t_n \rrbracket^M > \in \llbracket P \rrbracket^M$
If $\phi$ and $\psi$ are formulae, then, for any situation $M$,

b. $\llbracket \neg \phi \rrbracket^M = 1$ iff $\llbracket \phi \rrbracket^M = 0$

c. $\llbracket \phi \land \psi \rrbracket^M = 1$ iff $\llbracket \phi \rrbracket^M = 1$ and $\llbracket \psi \rrbracket^M = 1$

\[ \llbracket \phi \lor \psi \rrbracket^M = 1 \text{ iff } \llbracket \phi \rrbracket^M = 1 \text{ or } \llbracket \psi \rrbracket^M = 1 \]

\[ \llbracket \phi \rightarrow \psi \rrbracket^M = 1 \text{ iff } \llbracket \phi \rrbracket^M = 0 \text{ or } \llbracket \psi \rrbracket^M = 1 \]

d. If $\phi$ is a formula and $v$ is a variable, then, for any situation $s$,

\[ \llbracket \forall v \phi \rrbracket^M = 1 \text{ iff } \llbracket \phi[c/v] \rrbracket^M = 1 \text{ for all constants } c. \]

\[ \llbracket \exists v \phi \rrbracket^M = 1 \text{ iff } \llbracket [c/v] \phi \rrbracket^M = 1 \text{ for some constant } c. \]

Recall: $\phi[c/v]$ reads as
"the formula resulting from having the constant $c$ instead of the variable $v$ in $\phi$"

**QUESTION 7:** Let us take the situation $s$ depicted in (27). Let us take a language $PrL1$ such that: the constants $a$, $b$, and $c$ denote the individuals $\blacksquare$, $\bullet$ and $\blacklozenge$, respectively, the unary predicate $A$ denotes the set of individuals with a circle around, and the binary predicate $R$ denotes the relation encoded by the arrows.

(27)

```
   a   b  
  □   □   □
```

Determine the truth value of the following formulae of $PrL1$ in $s$, justifying it in detail.

(28)  a. $\exists x \exists y \exists z \left( R(x, y) \land A(y) \land R(x, z) \land A(z) \right)$

b. $\forall x \left( R(x, x) \right)$

c. $\forall x \left( R(x, x) \rightarrow A(x) \right)$

d. $\exists x \exists y \left( R(x, y) \land A(x) \land A(y) \right)$

**QUESTION 4:** What problem(s) do the semantic rules in (26d) present?

**Variable assignment – $g$:**

- a function from variables in the language to things they stand for in the model

- What are variables? Placeholder expressions:
  - Things that depend on context in a particular way:
(29) He runs. Mary saw them. She loves him.

  o Also a way to connect one part of expression with another:
(30) Everyone loves their mother.
(31) Every person is such that s/he runs. (Everyone runs)
(32) Some apple is such that it is red. (Some apple is red)

• So, variable assignment (in natural language) is sometimes a matter of extra-linguistic context (the person I’m pointing to, the people I just mentioned); and at other times just a device to help us figure out a meaning of an expression.
  o Just knowing how to understand English doesn’t help with interpreting words like ‘he’
  o So, the interpretation function I doesn’t help with variables (need assignments)

Take two:
(33) Variable assignments: functions g: set of variables → universe of individuals, De

\[ \llbracket g \rrbracket^M, g \]

(26') a. If \( \alpha \) is a constant (excluding syncategorematically treated symbols), then \( \llbracket \alpha \rrbracket^M, g \) is specified in the Lexicon for each situation.
  b. If \( \alpha \) is a variable, then \( \llbracket \alpha \rrbracket^M, g = g(\alpha) \)

(34) \( g_d/v \) reads as "the variable assignment \( g' \) that is exactly like \( g \) except (maybe) for \( g(v) \), which equals the individual \( d \)."

QUESTION 8: Complete the equivalences:
(35) \( g(x) = Mary \quad g_{Paul/x}(x) = \quad g_{Paul/x Susan/x}(x) = \quad g_{Paul/x Susan/y}(x) = \)
\( g(y) = Susan \quad g_{Paul/x}(y) = \quad g_{Paul/x Susan/x}(y) = \quad g_{Paul/x Susan/y}(y) = \)

(26') a. If \( P \) is a \( n \)-ary predicate and \( t_1, \ldots, t_n \) are all terms, then \( \llbracket P(t_1, \ldots, t_n) \rrbracket^M, g = 1 \) iff
\( < \llbracket t_1 \rrbracket^M, g, \ldots, \llbracket t_n \rrbracket^M, g > \in \llbracket P \rrbracket^M, g \)
  If \( \phi \) and \( \psi \) are formulae, then, for any situation,
  b. \( \llbracket \neg \phi \rrbracket^M, g = 1 \) iff \( \llbracket \phi \rrbracket^M, g = 0 \)
  c. \( \llbracket \phi \land \psi \rrbracket^M, g = 1 \) iff \( \llbracket \phi \rrbracket^M, g = 1 \) and \( \llbracket \psi \rrbracket^M, g = 1 \)
  \( \llbracket \phi \lor \psi \rrbracket^M, g = 1 \) iff \( \llbracket \phi \rrbracket^M, g = 1 \) or \( \llbracket \psi \rrbracket^M, g = 1 \)
  \( \llbracket \phi \rightarrow \psi \rrbracket^M, g = 1 \) iff \( \llbracket \phi \rrbracket^M, g = 0 \) or \( \llbracket \psi \rrbracket^M, g = 1 \)
  d. If \( \phi \) is a formula and \( v \) is a variable, then, for any situation,
\[ \forall \, v \, \phi \, \models_{M,g} = 1 \text{ iff } \models_{M, \phi}, \frac{d}{g} = 1 \text{ for all } d \in E. \]
\[ \exists \, v \, \phi \, \models_{M,g} = 1 \text{ iff } \models_{M, \phi}, \frac{d}{g} = 1 \text{ for some } d \in E. \]

(26”) For any formula \( \phi \), \( \models_{M} = 1 \) iff, for all assignments \( g \), \( \models_{M, g} = 1 \)

QUESTION 9: Take again the situation described in (27), repeated below. Determine the truth value of the same four formulae in this situation, explaining in detail how you followed the new strategy (“take two”) to find it out.

(27)

\[ \begin{array}{ccc}
\text{a} & \rightarrow & \text{b} \\
\downarrow & & \downarrow \\
\text{c} & & \\
\end{array} \]

Determine the truth value of the following formulae of PrL in s, justifying it in detail.

(28) a. \( \exists x \exists y \exists z \, ( R(x, y) \land A(y) \land R(x, z) \land A(z) ) \)

b. \( \forall x \, ( R(x, x) ) \)

c. \( \forall x \, ( R(x, x) \rightarrow A(x) ) \)

d. \( \exists x \exists y \, ( R(x, y) \land A(x) \land A(y) ) \)

5 Things to come

A new definition of logical inference: from syntactic (Natural Deduction) to semantic (model-theoretic) entailment.

Some important logical equivalences in PL and PrL.

Things we can do with this, in natural language.

References