

Notes for second semester algebraic topology

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Abstract

These are some notes for a second-semester algebraic topology course at UC Berkeley which I taught in 2004 and 2005. The topics of the course are: higher homotopy groups and obstruction theory, bundles and characteristic classes, spectral sequences, and Morse theory and applications to differential topology.

Disclaimers: These notes are intended as an elementary introduction to selected core ideas; most of this material is covered in much greater depth and generality in a number of standard texts. The current version of the notes is a bit rough.

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0 Prerequisites

The prerequisites for this course are the fundamental group, covering spaces, singular homology, cohomology, cup product, CW complexes, manifolds, and Poincaré duality.

Regarding covering spaces, recall the Lifting Criterion: if $\tilde{X} \xrightarrow{\pi} X$ is a covering space, if $f : (Y, y_0) \rightarrow (X, x_0)$, and if $\pi(\tilde{x}_0) = x_0$, then f lifts to a map $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ with $\pi \circ \tilde{f} = f$ if and only if

$$f_*\pi_1(Y, y_0) \subset \pi_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Regarding homology, there are many ways to define it. All definitions of homology that satisfy the Eilenberg-Steenrod axioms give the same answer for any reasonable space, e.g. any space homotopy equivalent to a CW complex. We will often use *cubical* singular homology, because this is convenient for some technical arguments. To define the homology of a space X , we consider “singular cubes”

$$\sigma : I^k \rightarrow X$$

where $I = [0, 1]$. The cube I^k has $2k$ codimension 1 faces

$$\begin{aligned} F_i^0 &= \{(t_1, \dots, t_k) \mid t_i = 0\}, \\ F_i^1 &= \{(t_1, \dots, t_k) \mid t_i = 1\}, \end{aligned}$$

for $1 \leq i \leq k$, each of which is homeomorphic to I^{k-1} in an obvious manner. We now define a boundary operator on formal linear combinations of singular cubes by

$$\partial\sigma := \sum_{i=1}^k (-1)^i (\sigma|_{F_i^0} - \sigma|_{F_i^1}).$$

This satisfies $\partial^2 = 0$. We could try to define $C_k(X)$ to be the free \mathbb{Z} -module generated by singular k -cubes and $H_k(X) = \text{Ker}(\partial) / \text{Im}(\partial)$. However this is not quite right because then the homology of a point would be \mathbb{Z} in every

nonnegative degree. To fix the definition, we mod out by the subcomplex of *degenerate* cubes that are independent of one of the coordinates on I^k , and then it satisfies the Eilenberg-Steenrod axioms.

Let us also recall an essential relation between cup product and intersections. Let M be a closed oriented smooth manifold of dimension n , and let A and B be closed oriented submanifolds of M , of codimension k and l respectively, intersecting transversely¹, so that the intersection $A \cap B$ is a closed oriented submanifold of codimension $k + l$. Let $[A] \in H_{n-k}(M)$ and $[B] \in H_{n-l}(M)$ denote the fundamental classes of A and B (more precisely the images of the fundamental classes under the inclusion into M). Let $[A]^* \in H^k(M; \mathbb{Z})$ and $[B]^* \in H^l(M; \mathbb{Z})$ denote their images under the Poincaré duality isomorphism

$$H_{n-*}(M) = H^*(M; \mathbb{Z}).$$

Then the fundamental class of the intersection $A \cap B$ is given in terms of the cup product by

$$[A \cap B]^* = [A]^* \smile [B]^* \in H^{k+l}(M; \mathbb{Z}). \quad (1)$$

Many interesting geometric questions can be phrased in terms of calculating the homology class of an intersection of submanifolds², and the relation (1) allows such questions to be answered in terms of the cup product.

¹We won't use too much differential topology, but it would be useful to know about transversality.

²Here is a nice example which I recommend learning. A version of the Lefschetz fixed point theorem says that if X is a closed smooth manifold and $f : X \rightarrow X$ is a smooth map with nondegenerate fixed points, then the signed count of the fixed points of f is given by

$$\# \text{Fix}(f) = \sum_i (-1)^i \text{Tr}(f_* : H_i(X; \mathbb{Q}) \rightarrow H_i(X; \mathbb{Q})).$$

(A fixed point p is *nondegenerate* if $1 - df_p : T_p X \rightarrow T_p X$ is invertible, and then the sign of the fixed point is the sign of $\det(1 - df_p)$.) This can be proved by interpreting the left hand side as the (signed) intersection number of the graph of f with the diagonal in $X \times X$ (these two submanifolds intersect transversely exactly when all fixed points are nondegenerate), using equation (1) to interpret this intersection number in terms of cup product, and then calculating the relevant cup product to obtain the right hand side.

1 Higher homotopy groups and obstruction theory

1.1 Higher homotopy groups

Let X be a topological space with a distinguished point x_0 . The fundamental group $\pi_1(X, x_0)$ has a generalization to *homotopy groups* $\pi_k(X, x_0)$, defined for each positive integer k .

The definition is simple. An element of $\pi_k(X, x_0)$ is a homotopy class of maps

$$f : (I^k, \partial I^k) \longrightarrow (X, x_0).$$

Here $I = [0, 1]$. Equivalently, an element of $\pi_k(X, x_0)$ is a homotopy class of maps $(S^k, p) \rightarrow (X, x_0)$ for some distinguished point $p \in S^k$.

The group operation is to stack two cubes together and then shrink:

$$fg(t_1, \dots, t_k) := \begin{cases} f(2t_1, t_2, \dots, t_k), & t_1 \leq 1/2, \\ g(2t_1 - 1, t_2, \dots, t_k), & t_1 \geq 1/2. \end{cases}$$

[Draw picture.] It is an exercise to check that $\pi_k(X, x_0)$ is a group, with the identity element given by the constant map sending I^k to x_0 .

A map $\phi : (X, x_0) \rightarrow (Y, y_0)$ defines an obvious map

$$\begin{aligned} \phi_* : \pi_k(X, x_0) &\longrightarrow \pi_k(Y, y_0), \\ f &\longmapsto f \circ \phi. \end{aligned}$$

This makes π_k a functor from pointed topological spaces to groups. Moreover, ϕ_* is clearly invariant under homotopy of ϕ , so a homotopy equivalence³ induces an isomorphism on π_k .

We also define $\pi_0(X)$ to be the set of path components of X , although this has no natural group structure.

Here are two nice properties of the higher homotopy groups.

Proposition 1.1 (a) $\pi_k(X \times Y, (x_0, y_0)) = \pi_k(X, x_0) \times \pi_k(Y, y_0)$.

(b) If $k > 1$, then $\pi_k(X, x_0)$ is abelian.

Proof. (a) Exercise. (b) [Draw picture] □

³Strictly speaking, this argument just works for a pointed homotopy equivalence, but we will see shortly that moving the basepoint induces an isomorphism on π_k .

Example 1.2 If $k > 1$, then $\pi_k(S^1, p) = 0$.

Proof. Recall that there is a covering space $\mathbb{R} \rightarrow S^1$. Any map $f : S^k \rightarrow S^1$ lifts to a map $\tilde{f} : S^k \rightarrow \mathbb{R}$, by the Lifting Criterion, since S^k is simply connected. Since \mathbb{R} is contractible, \tilde{f} is homotopic to a constant map. Projecting this homotopy to S^1 defines a homotopy of f to a constant map. \square

Example 1.3 More generally, the same argument shows that if the universal cover of X is contractible, then $\pi_k(X, x_0) = 0$ for all $k > 1$. For example, this holds if X is a Riemann surface of positive genus. This argument is a special case of the long exact sequence in homotopy groups of a fibration, which we will learn about later.

Example 1.4 We will prove shortly that⁴

$$\pi_k(S^n, x_0) \simeq \begin{cases} 0, & 1 \leq k < n, \\ \mathbb{Z}, & k = n. \end{cases} \quad (2)$$

The higher homotopy groups $\pi_k(S^n)$ for $k > n$ are very complicated, and despite extensive study are not completely understood, even for $n = 2$! This is the bad news about higher homotopy groups: despite their simple definition, they are generally hard to compute.⁵ However, we can compute at least some of them. This is important because as we will see, higher homotopy groups arise naturally, for example in obstruction theory.

1.2 The Hurewicz isomorphism theorem

To compute some higher homotopy groups, we begin by studying the relation between higher homotopy groups and homology. The key ingredient is the

⁴This can also be seen using some differential topology. Choose a point $x_1 \in S^n$ with $x_0 \neq x_1$. Let $f : (S^k, p) \rightarrow (S^n, x_0)$. By a homotopy we can arrange that f is smooth. Moreover if $k < n$ then we can arrange that $x_1 \notin f(S^k)$. Then f maps to $S^n \setminus \{x_1\} \simeq \mathbb{R}^n$, which is contractible, so f is homotopic to a constant map. If $k = n$ then we can arrange that f is transverse to x_1 so that x_1 has finitely many inverse images, to each of which is associated a sign. It can then be shown that the signed count $\#f^{-1}(x_1) \in \mathbb{Z}$ determines the homotopy class of f in $\pi_k(S^k, x_0)$.

⁵Recall that the Seifert-Van Kampen theorem gives an algorithm for computing, or at least finding a presentation of, $\pi_1(X, x_0)$ whenever X can be cut into simple pieces. The idea is that a loop in X can be split into paths which live in the various pieces. This fails for higher homotopy groups because if one attempts to cut a sphere into pieces, then these pieces might be much more complicated objects.

Hurewicz homomorphism

$$\Phi : \pi_k(X, x_0) \longrightarrow H_k(X),$$

defined as follows. Recall that once we choose an orientation for S^k , there is a canonical isomorphism

$$H_k(S^k) \simeq \mathbb{Z}.$$

The generator is the *fundamental class* $[S^k] \in H_k(S^k)$. If $f : (S^k, p) \rightarrow (X, x_0)$ represents $[f] \in \pi_k(X, x_0)$, we define

$$\Phi[f] := f_*[S_k] \in H_k(X).$$

Alternatively, if we use cubical singular homology, then a map $f : (I^k, \partial I^k) \rightarrow (X, x_0)$, regarded as a singular cube, defines a cycle in the homology class $\Phi[f]$. By the homotopy invariance of homology, $\Phi[f]$ is well-defined, i.e. depends only on the homotopy class of f . It is an exercise to check that Φ is a homomorphism.

Also, it follows immediately from the definition that the Hurewicz map Φ is natural, in the following sense: If $\psi : (X, x_0) \rightarrow (Y, y_0)$, then the diagram

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{\Phi} & H_k(X) \\ \downarrow \psi_* & & \downarrow \psi_* \\ \pi_k(Y, y_0) & \xrightarrow{\Phi} & H_k(Y) \end{array} \quad (3)$$

commutes. That is, Φ is a natural transformation of functors from π_k to H_k .

Recall now that if X is path connected, then Φ induces an isomorphism

$$\frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]} \simeq H_1(X).$$

This fact has the following generalization, asserting that if X is also simply connected, then the first nontrivial higher homotopy group is isomorphic to the first nontrivial reduced homology group, and implying equation (2) for the first nontrivial homotopy groups of spheres.

Theorem 1.5 (Hurewicz isomorphism theorem) *Let $k \geq 2$. Suppose that X is path connected and that $\pi_i(X, x_0) = 0$ for all $i < k$. Then the Hurewicz map induces an isomorphism*

$$\pi_k(X, x_0) \simeq H_k(X).$$

To prove this theorem, we will need the following useful facts about homotopy groups. Below, $F_j I^k$ denotes the set of j -dimensional faces of the k -cube.

Lemma 1.6 *Let (X, x_0) be a pointed space.*

- (a) *For any $k \geq 1$, if $f : (S^k, p) \rightarrow (X, x_0)$ is homotopic, without fixing the base points, to a constant map, then $[f] = 0 \in \pi_k(X, x_0)$.*
- (b) *For any $k \geq 2$, let $f : \partial I^{k+1} \rightarrow X$ be a map sending every $(k-1)$ -dimensional face to x_0 . Then*

$$[f] = \sum_{\sigma \in F_k(I^{k+1})} [f|_{\sigma}] \in \pi_k(X, x_0).$$

Proof. We will just sketch the argument and leave the details as an exercise.

(a) Given a homotopy $\{f_t\}$ with $f_0 = f$ and f_1 constant, one can use the trajectory of p , namely the path $t \mapsto f_t(p)$, to modify this to a homotopy sending p to x_0 at all times. (Note that more generally, if $f, g : (S^k, p) \rightarrow (X, x_0)$ are homotopic without fixing the base points, it does *not* follow that $[f] = [g] \in \pi_k(X, x_0)$. Compare §1.3.)

(b) We use a homotopy to shrink the restrictions of f to the k -dimensional faces of I^{k+1} , so that most of ∂I^{k+1} is mapped to x_0 . Identifying $\partial I^{k+1} \simeq S^k$, and using the fact that $k \geq 3$, we can then move around these k -cubes until they are lined up as in the definition of composition in π_k . \square

*Proof of the Hurewicz isomorphism theorem.*⁶ Using singular homology with cubes, we define a map

$$\Psi : C_k(X) \longrightarrow \pi_k(X, x_0)$$

as follows. The idea is that a generator of $C_k(X)$ is a cube whose boundary may map anywhere in X , and we have to modify it, via a chain homotopy, to obtain a cube whose boundary maps to x_0 .

Since X is path connected, for each 0-cube $p \in X$, we can choose a path $K(p)$ from x_0 to p . For each 1-cube $\sigma : I \rightarrow X$, there is a map $\partial I^2 \rightarrow X$

⁶Later we will see a shorter and slicker proof, which uses the machinery of spectral sequences, but which I think ultimately has the same underlying geometric content.

sending the four faces to x_0 , σ , $K(\sigma(0))$, and $K(\sigma(1))$. Since X is simply connected, this can be extended to a map $K(\sigma) : I^2 \rightarrow X$ such that

$$\partial K(\sigma) = \sigma - K(\partial\sigma). \quad (4)$$

Continuing by induction on i , if $1 \leq i < k$, then for each i -cube $\sigma : I^i \rightarrow X$, we can choose⁷ an $(i+1)$ -cube $K(\sigma) : I^{i+1} \rightarrow X$ which sends the faces to x_0 , σ , and the faces of $K(\partial\sigma)$, and therefore satisfies equation (4). Finally, if $\sigma : I^k \rightarrow X$ is a k -cube, then there is a map $\partial I^{k+1} \rightarrow X$ sending the faces to x_0 , σ , and the faces of $K(\partial\sigma)$. Let F denote the face sent to x_0 . Identifying $(\partial I^{k+1}/F, F) \simeq (I^k, \partial I^k)$, this gives an element $\Psi(\sigma) \in \pi_k(X, x_0)$. Moreover, it is easy to see that the sum of the faces other than F is homologous to $\Psi(\sigma)$, i.e. there is a cube $K(\sigma)$ with

$$\partial K(\sigma) = \sigma - \Psi(\sigma) - K\partial\sigma. \quad (5)$$

While Ψ may depend on the above choices, we claim that Ψ induces a map on homology which is inverse to Φ . To start, we claim that if $\sigma : I^{k+1} \rightarrow X$ is a $(k+1)$ -cube, then

$$\Psi(\partial\sigma) = 0. \quad (6)$$

To see this, first note that by part (b) of the lemma, $\Psi(\partial\sigma) = [f]$, where $f : \partial I^{k+1} \rightarrow X$ sends each k -dimensional face of I^{k+1} to Ψ of the corresponding face of σ . By cancelling stuff along adjacent faces, f is homotopic, without fixing base points, to $\sigma|_{\partial I^{k+1}}$. This is homotopic to a constant map since it extends over I^{k+1} . So by part (a) of the lemma, $[f] = 0$. It follows from (6) that Ψ induces a map $\Psi_* : H_k(X) \rightarrow \pi_k(X, x_0)$.

We can make the above choices so that

- (*) If $i < k$ and $\sigma : I^i \rightarrow X$ is a constant map to x_0 , then $K(\sigma)$ is also a constant map to x_0 .

It is then apparent that $\Psi_* \circ \Phi = \text{id}_{\pi_k(X, x_0)}$.

To complete the proof, we must show that $\Phi \circ \Psi_* = \text{id}_{H_k(X)}$. By equation (5), any k -dimensional homology class can be represented by a sum of k -cubes $\sigma : I^k \rightarrow X$ that send $\partial I^k \rightarrow x_0$. Each such cube is itself a cycle. So we just need to show that if σ is such a cube, then $\Phi(\Psi(\sigma)) = [\sigma] \in H_k(X)$. This follows easily from the condition (*). \square

⁷It's not really necessary to make infinitely many choices in this proof, but the argument is less awkward this way. Incidentally, when σ is degenerate, we should choose $K(\sigma)$ to be degenerate as well.

1.3 Dependence of π_k on the base point

We will now show that if X is path connected, then the homotopy groups of X for different choices of base point are isomorphic, although not always equal. This is perhaps a bit nitpicky, but worth understanding properly since similar structures commonly arise elsewhere in mathematics.

If $\gamma : [0, 1] \rightarrow X$ is a path from x_0 to x_1 , we define a map

$$\Phi_\gamma : \pi_k(X, x_1) \longrightarrow \pi_k(X, x_0)$$

as follows. For now let us parametrize the k -cube as $I^k = [-1, 1]^k$. If $f : [-1, 1]^k \rightarrow X$ and $t = (t_1, \dots, t_k) \in I^k$, let $m = \max\{|t_i|\}$ and define

$$\Phi_\gamma(f)(t) := \begin{cases} f(2t), & m \leq 1/2, \\ \gamma(2(1-m)), & m \geq 1/2. \end{cases}$$

[Draw picture.] Clearly this gives a well-defined function on homotopy groups. It is an exercise to show that Φ_γ is a group homomorphism.

The following additional facts are easy to see:

(i) If γ is homotopic to γ' (rel endpoints), then

$$\Phi_\gamma = \Phi_{\gamma'} : \pi_k(X, x_1) \longrightarrow \pi_k(X, x_0).$$

(ii) Suppose that γ_1 is a path from x_0 to x_1 and γ_2 is a path from x_1 to x_2 , and let $\gamma_1\gamma_2$ denote the composite path. Then

$$\Phi_{\gamma_1\gamma_2} = \Phi_{\gamma_1}\Phi_{\gamma_2} : \pi_k(X, x_2) \longrightarrow \pi_k(X, x_0).$$

(iii) If γ is the constant path at x_0 , then

$$\Phi_\gamma = \text{id}_{\pi_k(X, x_0)}.$$

The above three properties⁸ imply:

⁸A fancy way of describing these three properties is that $\pi_k(X, \cdot)$ is a functor from the *fundamental groupoid* of X to groups. Alternatively, $\pi_k(X, \cdot)$ is a “twisted coefficient system” on X ; these will be important to us later.

Proposition 1.7 *If X is path connected, then*

$$\pi_k(X, x_0) \simeq \pi_k(X, x_1)$$

for any two points $x_0, x_1 \in X$. Moreover, if X is simply connected, then this isomorphism is canonical, and so $\pi_k(X)$ is a well-defined group without the choice of a base point.

If X is not simply connected, then the isomorphism is not canonical, in which case $\pi_k(X)$ is not well-defined without the choice of a base point. That is, for a noncontractible loop γ based at x_0 , the isomorphism Φ_γ of $\pi_k(X, x_0)$ might not be the identity. In general, all we can say is that there is an action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$. When $k = 1$, it is easy to see that this action is conjugation, which of course is nontrivial whenever π_1 is not abelian.

We now construct, for arbitrary $k > 1$, examples of spaces in which the action of π_1 on π_k is nontrivial. Let X be any space and let $f : X \rightarrow X$ be a homeomorphism. The *mapping torus* of f is the quotient space

$$Y_f := \frac{X \times [0, 1]}{(x, 1) \sim (f(x), 0)}.$$

For example, if $X = [-1, 1]$ and f is multiplication by -1 , then Y_f is the Möbius band. If f is the identity, then $Y_f = X \times S^1$.

There is a natural inclusion $\iota : X \rightarrow Y_f$ sending $x \mapsto (x, 0)$. Pick a base point $x_0 \in X$, and let $y_0 = \iota(x_0)$.

Lemma 1.8 *For $k \geq 2$, the inclusion induces an isomorphism*

$$\iota_* : \pi_k(X, x_0) \xrightarrow{\simeq} \pi_k(Y_f, y_0).$$

Proof. The mapping torus can equivalently be defined as

$$Y_f = \frac{X \times \mathbb{R}}{(x, t+1) \sim (f(x), t)}.$$

Hence $\mathbb{R} \times X$ is a covering space of Y_f . By the Lifting Criterion, since $k \geq 2$, a map $(S^k, p) \rightarrow (Y_f, y_0)$ lifts to a map $(S^k, p) \rightarrow (X \times \mathbb{R}, (x_0, 0))$. Since \mathbb{R} is contractible, this map is homotopic to map $(S^k, p) \rightarrow (X \times \{0\}, (x_0, 0))$. Projecting this homotopy back down to Y_f shows that ι_* is surjective. Applying the same argument to homotopies $S^k \times I \rightarrow Y_f$ shows that ι_* is injective.

(Compare Example 1.2, and the long exact sequence of a fibration to be introduced later.) \square

Now suppose that x_0 is a fixed point of f . This defines a loop γ in Y_f based at x_0 , sending $t \mapsto (x_0, t)$. It is an exercise to show that the corresponding isomorphism of $\pi_k(Y_f, y_0)$ is given by (the inverse of) f_* , i.e.:

Proposition 1.9 *The following diagram commutes:*

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{f_*^{-1}} & \pi_k(X, x_0) \\ \iota_* \downarrow & & \iota_* \downarrow \\ \pi_k(Y, y_0) & \xrightarrow{\Phi_\gamma} & \pi_k(Y, y_0). \end{array}$$

Example 1.10 To get an explicit example where Φ_γ is nontrivial, let $X = S^k$, and let $f : S^k \rightarrow S^k$ be a degree -1 homomorphism with a fixed point x_0 . By the naturality of the Hurewicz isomorphism (3), $f_* = -1$ on $\pi_k(S^k) = \mathbb{Z}$.

Exercise 1.11 If X is path connected, then there is a canonical bijection

$$[S^k, X] = \pi_k(X, x_0) / \pi_1(X, x_0).$$

Here the left hand side denotes the space of free (no basepoints) homotopy classes of maps from S^k to X , and the right hand side denotes the quotient of $\pi_k(X, x_0)$ by the action of $\pi_1(X, x_0)$.

1.4 Fiber bundles

A fiber bundle is a kind of “family” of topological spaces. These are important objects of study in topology and in this course, and also will help us compute homotopy groups.

Definition 1.12 A *fiber bundle*⁹ is a map of topological spaces $\pi : E \rightarrow B$ such that there exists a topological space F with the following property:

⁹Different authors might define this term differently.

(Local triviality) For all $x \in B$, there is a neighborhood U of x in B , and a homeomorphism $\pi^{-1}(U) \simeq U \times F$, such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\simeq} & U \times F \\ \pi \downarrow & & \downarrow \\ U & \xrightarrow{=} & U \end{array}$$

commutes, where the map $U \times F \rightarrow U$ is projection on to the first factor.

In particular, for each $x \in B$, we have $\pi^{-1}(x) \simeq F$. Thus the fiber bundle can be thought of as a family of topological spaces, each homeomorphic to F , and parametrized by B . One calls F the *fiber*, E the *total space*, and B the *base space*. A fiber bundle with fiber F and base B is an “ F -bundle over B ”. Also, $\pi^{-1}(x)$ is the *fiber over x* , denoted by E_x . We denote the fiber bundle by the diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B, \end{array}$$

or sometimes just by E .

Definition 1.13 If $\pi_1 : E_1 \rightarrow B$ and $\pi_2 : E_2 \rightarrow B$ are fiber bundles over the same base space, then a morphism¹⁰ of fiber bundles from π_1 to π_2 is a map $f : E_1 \rightarrow E_2$ such that $\pi_2 \circ f = \pi_1$.

Example 1.14 For any B and F , we have the *trivial bundle* $E = B \times F$, with $\pi(b, f) = b$.

The local triviality condition says that any fiber is locally isomorphic to a trivial bundle. We will also prove shortly that:

Example 1.15 If B is a contractible CW complex, then any fiber bundle over B is trivial.

However, when B has some nontrivial topology, fiber bundles can have some global “twisting”.

¹⁰Different definitions are possible. A less restrictive definition allows the base spaces to be different. A more restrictive definition requires the map to be a homeomorphism on each fiber.

Example 1.16 A covering space is a fiber bundle in which the fiber F has the discrete topology.

Example 1.17 Let $f : X \rightarrow X$ be a homeomorphism. Then the mapping torus Y_f is the total space of a fiber bundle

$$\begin{array}{ccc} X & \longrightarrow & Y_f \\ & & \downarrow \pi \\ & & S^1 \end{array}$$

where $\pi(x, t) = t \pmod{1}$. Exercise: every fiber bundle over S^1 arises in this way.

The previous example has the following generalization, called the “clutching construction”. Let $\text{Homeo}(F)$ denote the space of homeomorphisms¹¹ of F , let $k > 1$, and let $\phi : S^{k-1} \rightarrow \text{Homeo}(F)$. Identify

$$S^k = D^k \cup_{S^{k-1}} D^k,$$

and define a fiber bundle over S^k by

$$E = (D^k \times F) \cup_{S^{k-1} \times F} (D^k \times F),$$

where the two copies of $D^k \times F$ are glued together along $S^{k-1} \times F$ by the map

$$\begin{aligned} S^{k-1} \times F &\longrightarrow S^{k-1} \times F, \\ (x, f) &\longmapsto (x, \phi(x)(f)). \end{aligned}$$

It follows from Example 1.15, applied to $B = D^k$, that every fiber bundle over S^k arises this way. Conversely, the fiber bundle depends only on the homotopy class of ϕ .

¹¹We topologize the space $\text{Maps}(X, Y)$ of continuous maps from X to Y using the compact-open topology. For details on this see e.g. Hatcher. The important property of this topology is that if Y is locally compact, then a map $X \rightarrow \text{Maps}(Y, Z)$ is continuous iff the corresponding map $X \times Y \rightarrow Z$ is continuous.

Example 1.18 S^1 -bundles over S^2 are classified by nonnegative integers¹². To see this, note that $\text{Homeo}(S^1)$ deformation retracts¹³ onto $O(2)$. Given an S^1 -bundle over S^2 , one can arrange that ϕ maps to $SO(2)$, by changing the identification of one of the halves $\pi^{-1}(D^2)$ with $D^2 \times S^1$, via id_{D^2} cross an orientation-reversing homeomorphism of S^1 . The nonnegative integer associated to the bundle is then $|\deg(\phi)|$; we will temporarily call this the “rotation number”¹⁴. Note that switching the identification of both halves will change the sign of $\deg(\phi)$.

Example 1.19 The *Hopf fibration*

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

is given by the map

$$S^3 \rightarrow \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}P^1 \simeq S^2.$$

Exercise 1.20 The Hopf fibration has “rotation number” 1. The unit tangent bundle of S^2 has “rotation number” 2.

Definition 1.21 A *section* of a fiber bundle $\pi : E \rightarrow B$ is a map $s : B \rightarrow E$ such that $\pi \circ s = \text{id}_B$.

Intuitively, a section is a continuous choice of a point in each fiber.

Example 1.22 A section of a trivial bundle $E = B \times F$ is just a map $s : B \rightarrow F$.

Nontrivial bundles may or may not possess sections. We will study this question systematically a little later. For now you can try to convince yourself that an S^1 -bundle over S^2 with nonzero rotation number does not have a section.

¹²It is more common to consider *oriented* S^1 -bundles over S^2 , which are classified by integers. We will discuss these later.

¹³This is a good exercise. More generally, one can ask whether $\text{Homeo}(S^k)$ deformation retracts to $O(k+1)$. This is known to be true for $k = 2$ and $k = 3$, the latter result being a consequence of the “Smale conjecture” proved by Hatcher (asserting that the inclusion $SO(4) \rightarrow \text{Diff}(S^3)$ is a homotopy equivalence) and a result of Cerf. I think it’s not true for all k but I haven’t chased down the references.

¹⁴This is really the absolute value of the Euler class for a choice of orientation of the bundle; we will discuss this later.

1.5 Homotopy properties of fiber bundles

We will now show, roughly, that the fiber bundles over a CW complex B depend only on the homotopy type of B . We will use the following terminology: if E is a fiber bundle over B , and if $U \subset B$, then a “trivialization of E over U ” is a homeomorphism $\phi : \pi^{-1}(U) \simeq U \times F$ commuting with the projections.

Lemma 1.23 *Any fiber bundle E over $B = I^k$ is trivial.*

Proof. By the local triviality condition and compactness of I^k , we can subdivide I^k into N^k subcubes of size $1/N$, such that E is trivial over each subcube. We now construct a trivialization of E over the whole cube, one subcube at a time. By an induction argument which we leave as an exercise [draw picture], it is enough to prove the following lemma. \square

Lemma 1.24 *Let $X = I^k$ and let $A \subset I^k$ be the union of all but one of the $(k - 1)$ -dimensional faces of I^k . Suppose that E is a trivial fiber bundle over X . Then a trivialization of E over A extends to a trivialization of E over X .*

Proof. We can regard $E = X \times F$. The trivialization of E over A can then be regarded as a map $\phi : A \rightarrow \text{Homeo}(F)$, and we have to extend this to a map $\bar{\phi} : X \rightarrow \text{Homeo}(F)$. Such an extension exists because there is a retraction $r : X \rightarrow A$, so that we can define $\bar{\phi} = \phi \circ r$. \square

Before continuing, we need to introduce the notion of pullback bundles. As a warmup, if E is a fiber bundle over B and if $A \subset B$, then the restriction of E to A is the fiber bundle $E|_A = \pi^{-1}(A)$ with base A .

More generally, if $f : B' \rightarrow B$, then the *pullback bundle* f^*E over B' is defined by¹⁵

$$f^*E := \{(x, y) \mid x \in B', y \in E, f(x) = \pi(y)\} \subset B' \times E.$$

¹⁵That is, f^*E is the *fiber product* of f and π over B' . There is a commutative diagram

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B. \end{array}$$

The map $f^*E \rightarrow B'$ induced by the projection $B' \times E \rightarrow B'$. The definition implies that the fiber of f^*E over $x \in B'$ is equal to the fiber of E over $f(x) \in B$.

Example 1.25 If $f = \text{id}_B$, then $f^*E = E$.

Example 1.26 If f is a constant map to $x \in B$, then f^*E is the trivial bundle

$$f^*E = B' \times E_x.$$

Example 1.27 If E is the mapping torus of $\phi : X \rightarrow X$, and if $f : S^1 \rightarrow S^1$ has degree d , then f^*E is the mapping torus of ϕ^d .

Similarly, we will see that if E is an S^1 -bundle over S^2 with rotation number r , and if $f : S^2 \rightarrow S^2$ has degree d , then f^*E is an S^1 -bundle over S^2 with rotation number $|dr|$.

Lemma 1.28 *Let B be a CW complex and let E be a fiber bundle over $B \times [0, 1]$. Then*

$$E|_{B \times \{0\}} \simeq E|_{B \times \{1\}},$$

where both sides are regarded as fiber bundles over B .

Proof. Let $f : B \times [0, 1] \rightarrow B$ be the projection. We will construct an isomorphism

$$f^*(E|_{B \times \{0\}}) \simeq E. \tag{7}$$

Restricting this isomorphism to $B \times \{1\}$ then implies the lemma.

We define the isomorphism (7) to be the identity over $B \times \{0\}$. We now extend this isomorphism over $B \times [0, 1]$, one cell (in B) at a time¹⁶, by induction on the dimension. This means that without loss of generality, $B = D^k$, and the isomorphism (7) has already been defined over $D^k \times \{0\}$ and $\partial D^k \times [0, 1]$. By Lemma 1.23, E and $f^*(E|_{B \times \{0\}})$ are trivial when $B = D^k \times [0, 1]$, so after choosing a trivialization, the desired isomorphism (7) is equivalent to a map $D^k \times [0, 1] \rightarrow \text{Homeo}(F)$. As in Lemma 1.24, such a map from $D^k \times \{0\} \cup \partial D^k \times [0, 1]$ can be extended over $D^k \times [0, 1]$. \square

¹⁶This is legitimate because a map from a CW complex is continuous if and only if it is continuous on each closed cell.

Proposition 1.29 *Let B' be a CW complex, let E be a fiber bundle over B , and let $f_0, f_1 : B' \rightarrow B$ be homotopic maps. Then*

$$f_0^*E \simeq f_1^*E$$

as fiber bundles over B' .

Proof. The homotopy can be described by map

$$f : B' \times [0, 1] \rightarrow B$$

with $f|_{B \times \{i\}} = f_i$. Then

$$f_0^*E = (f^*E)|_{B \times \{0\}} \simeq (f^*E)|_{B \times \{1\}} = f_1^*E,$$

where the middle isomorphism holds by Lemma 1.28. \square

Corollary 1.30 *If B is a contractible CW complex, then any fiber bundle over B is trivial.*

Proof. Contractibility means that there is a point $x \in B$ and a homotopy between id_B and the constant map $f : B \rightarrow \{x\}$. So by Proposition 1.29, if E is a fiber bundle over B , then

$$E = \text{id}_B^*E \simeq f^*E = B \times E_x.$$

\square

More generally, a homotopy equivalence between B_1 and B_2 induces a bijection between isomorphism classes of F -bundles over B_1 and B_2 .

1.6 A long exact sequence in homotopy groups

Theorem 1.31 *Let $F \rightarrow E \rightarrow B$ be a fiber bundle. Pick $y_0 \in E$, let $x_0 = \pi(y_0) \in B$, and identify $F = \pi^{-1}(x_0)$. Then there is a long exact sequence*

$$\cdots \rightarrow \pi_{k+1}(B, x_0) \rightarrow \pi_k(F, y_0) \rightarrow \pi_k(E, y_0) \rightarrow \pi_k(B, x_0) \rightarrow \pi_{k-1}(F, y_0) \rightarrow \cdots \quad (8)$$

The exact sequence terminates at $\pi_1(B, x_0)$. The arrows $\pi_k(F, y_0) \rightarrow \pi_k(E, y_0)$ and $\pi_k(E, y_0) \rightarrow \pi_k(B, x_0)$ are induced by the maps $F \rightarrow E$ and $E \rightarrow B$ in the fiber bundle. The definition of the connecting homomorphism $\pi_k(B, x_0) \rightarrow \pi_{k-1}(F, y_0)$ and the proof of exactness will be given later. Let us first do some computations with this exact sequence.

Example 1.32 If $E \rightarrow B$ is a covering space, then it follows from the exact sequence that $\pi_k(E, y_0) \simeq \pi_k(B, x_0)$ for all $k \geq 2$. This generalizes some examples we considered previously.

Example 1.33 Applying the exact sequence to the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$, we get an exact sequence

$$\pi_3(S^1) \rightarrow \pi_3(S^3) \rightarrow \pi_3(S^2) \rightarrow \pi_2(S^1).$$

We know that this is $0 \rightarrow \mathbb{Z} \rightarrow \pi_3(S^2) \rightarrow 0$, and therefore

$$\pi_3(S^2) \simeq \mathbb{Z}. \tag{9}$$

Remark 1.34 Just for fun, we can describe the isomorphism (9) explicitly as follows. Any homotopy class in $\pi_3(S^2)$ can be represented by a smooth map $f : S^3 \rightarrow S^2$. By Sard's theorem, for a generic point $p \in S^2$, the map f is transverse to p , so that the inverse image $f^{-1}(p)$ is a smooth link in S^3 . This link inherits an orientation from the orientations of S^3 and S^2 . Let q be another generic point in S^2 , and define the *Hopf invariant* of f to be the linking number¹⁷ of $f^{-1}(p)$ and $f^{-1}(q)$:

$$H(f) := \ell(f^{-1}(p), f^{-1}(q)) \in \mathbb{Z}.$$

The integer H is a homotopy invariant of f and defines the isomorphism (9). This is a special case of the Thom-Pontrjagin construction, which we may discuss later.

The long exact sequence can be used to compute a few homotopy groups of Lie groups, by the use of suitable fiber bundles. For example, let us try to compute some homotopy groups of $SU(n)$. There is a map $SU(n) \rightarrow S^{2n-1}$ which sends $A \mapsto Av$, where v is some fixed unit vector.

Exercise 1.35 This gives a fiber bundle $SU(n-1) \rightarrow SU(n) \rightarrow S^{2n-1}$.

Note that $SU(2) \simeq S^3$. So the first interesting case of this is when $n = 3$, and we get

$$S^3 \rightarrow SU(3) \rightarrow S^5.$$

¹⁷If you don't know what linking number is, then put aside this homotopy theory stuff right now and go learn about it.

So by the long exact sequence, $\pi_1 SU(3) = \pi_2 SU(3) = 0$, and $\pi_3 SU(3) = \mathbb{Z}$. In general, while the long exact sequence of this fiber bundle does not allow us to compute $\pi_k(SU(n))$ in all cases, it does show that $\pi_k(SU(n))$ is independent of n when n is sufficiently large with respect to k . This group is denoted by $\pi_k(SU)$ and its value is given by Bott Periodicity, which we may discuss later.

We now construct the long exact sequence. It works in a more general context than that of fiber bundles, which we now define.

Definition 1.36 A map $\pi : E \rightarrow B$ has the *homotopy lifting property* with respect to a pair (X, A) if given maps $f : X \times I \rightarrow B$ and $g : X \times \{0\} \cup A \times I \rightarrow E$ with $\pi \circ g = f|_{X \times \{0\} \cup A \times I}$, there exists $h : X \times I \rightarrow E$ extending g such that $\pi \circ h = f$.

Example 1.37 Recall that if $\pi : E \rightarrow B$ is a covering space, then any path in B lifts to a path in E , given a lift of its initial endpoint. This means that $\pi : E \rightarrow B$ has the homotopy lifting property with respect to the pair (pt, \emptyset) .

Definition 1.38 A map $\pi : E \rightarrow B$ is a *Serre fibration* if it has the homotopy lifting property with respect to the pair $(I^k, \partial I^k)$ for all k .

Lemma 1.39 *Every fiber bundle is a Serre fibration.*

Proof. Let $(X, A) = (I^k \times I, I^k \times \{0\} \cup \partial I^k \times I)$, let $\pi : E \rightarrow B$ be a fiber bundle, and let $f : X \rightarrow B$. We are given a lift g to E of the restriction of f to A , and we must extend this to a lift of f over all of X . Now g is equivalent to a section of $f^*E|_A$, and the problem is to extend this section over X . We know that f^*E is trivial, so after choosing a trivialization, a section is equivalent to a map to F . But a map $A \rightarrow F$ extends over X because there is a retraction $r : X \rightarrow A$. \square

Exercise 1.40 Let $E = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\}$, let $B = I$, and let $\pi : E \rightarrow B$ send $(x, y) \mapsto x$. Then π is a Serre fibration, but not a fiber bundle.

Exercise 1.41 A map $\pi : E \rightarrow B$ is a *fibration* if it has the homotopy lifting property with respect to (X, ϕ) for all spaces X .

(a) Show that a fibration is a Serre fibration.

- (b) Show that all the fibers of a fibration over a path connected space B are homotopy equivalent.
- (c) Show that all the fibers of a Serre fibration over a path connected space B have isomorphic homotopy and homology groups.

Theorem 1.42 *If $\pi : E \rightarrow B$ is a Serre fibration, then there is a long exact sequence in homotopy groups (8).*

Proof. For $k \geq 1$ we need to define a connecting homomorphism

$$\delta : \pi_{k+1}(B, x_0) \longrightarrow \pi_k(F, y_0).$$

Consider an element of $\pi_{k+1}(B, x_0)$ represented by a map $f : I^{k+1} \rightarrow B$ sending ∂I^{k+1} to x_0 . We can lift f to E over $I^k \times \{0\} \cup \partial I^k \times I$, by mapping that set to y_0 . By the homotopy lifting property, this lift extends to a lift $h : I^k \times I \rightarrow E$ of f . Observe that $h|_{I^k \times \{1\}}$, regarded as a map on I^k , sends $(I^k, \partial I^k) \rightarrow (F, y_0)$, because f sends $I^k \times \{1\}$ to x_0 . We now define

$$\delta[f] := [h|_{I^k \times \{1\}}].$$

One can check that this is well-defined. With this definition, the proof of exactness is a more or less straightforward exercise using the homotopy lifting property. \square

Exercise 1.43 Let E be an S^1 -bundle over S^2 with rotation number r . Then the connecting homomorphism

$$\delta : \mathbb{Z} \simeq \pi_2(S^2) \rightarrow \pi_1(S^1) \simeq \mathbb{Z}$$

is multiplication by $\pm r$.

Example 1.44 Let X be a topological space and $x_0 \in X$. The *based loop space* $\Omega X = \{\gamma : [0, 1] \rightarrow X \mid \gamma(0) = \gamma(1) = x_0\}$. Define the space of paths starting at x_0 to be $PX = \{\gamma : [0, 1] \rightarrow X \mid \gamma(0) = x_0\}$. This fits into the *path fibration*

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \downarrow \pi \\ & & X \end{array}$$

where $\pi(f) = f(1)$. Now PX is contractible, so the long exact sequence gives

$$\pi_k(\Omega X) \simeq \pi_{k+1}(X)$$

for $k \geq 1$. This can also be seen directly from the definition. But one can get a lot more information out of the path fibration as we will see later.

1.7 Obstruction theory: first examples

The rough idea of obstruction theory is simple. Suppose we want to construct some kind of function on a CW complex X . We do this by induction: if the function is defined on the k -skeleton X^k , we try to extend it over the $(k+1)$ -skeleton X^{k+1} . The obstruction to extending over a $(k+1)$ -cell is an element of π_k of something. These obstructions fit together to give a cellular cochain \mathfrak{o} on X with coefficients in this π_k . In fact this cochain is a cocycle, so it defines an “obstruction class” in $H^{k+1}(X; \pi_k(\text{something}))$. If this cohomology class is zero, i.e. if there is a cellular k -cochain η with $\mathfrak{o} = \delta\eta$, then η prescribes a way to modify our map over the k -skeleton so that it can be extended over the $(k+1)$ -skeleton.

Let us now make this idea more precise by proving some theorems. For now, if X is a CW complex, $C_*(X)$ will denote the cellular chain complex. We assume that each cell has an orientation chosen, so that $C_k(\mathbb{Z})$ is the free \mathbb{Z} -module generated by the k -cells. If e is a k -cell and e' is a $(k-1)$ -cell, we let $\langle \partial e, e' \rangle \in \mathbb{Z}$ denote the coefficient of e' in ∂e .

Theorem 1.45 *Let X be a CW complex. Then*

$$[X, S^1] = H^1(X; \mathbb{Z}).$$

Proof. Let α be the preferred generator of $H^1(S^1; \mathbb{Z}) = \mathbb{Z}$. Define a map

$$\begin{aligned} \Phi : [X, S^1] &\longrightarrow H^1(X; \mathbb{Z}), \\ [f : X \rightarrow S^1] &\longmapsto f^*\alpha. \end{aligned}$$

We want to show that Φ is a bijection.

Proof that Φ is surjective: let $\xi \in C^1(X; \mathbb{Z})$ with $\delta\xi = 0$. We need to find $f : X \rightarrow S^1$ with $f^*\alpha = [\xi]$. We construct f on the k skeleton by induction on k . First, we send the 0-skeleton to a base point $p \in S^1$. Next, if σ is a one-cell, we extend f over σ so that $f|_\sigma$ has winding number $\xi(\sigma)$ around S^1 .

If $e : D^2 \rightarrow X$ is a 2-cell, then since ξ is a cocycle,

$$\sum_{\sigma} \langle \partial e, \sigma \rangle \xi(\sigma) = 0.$$

it is not hard to see (and we will prove a more general statement in Lemma 1.48 below) that the left hand side of the above equation is the winding number of $f \circ e|_{\partial D^2}$ around S^1 . Hence, we can extend f over the 2-cell.

Now assume that f has been defined over the k -skeleton X^k for some $k \geq 2$. We can then extend f over the $(k+1)$ -skeleton, because the obstruction to extending over any $(k+1)$ -cell lives in $\pi_k(S^1) = 0$.

Proof that Φ is injective: Let $f_0, f_1 : X \rightarrow S^1$, and suppose that $f_0^* \alpha = f_1^* \alpha$. We can homotope¹⁸ f_0 and f_1 so that they send the 0-skeleton of X to the base point $p \in S^1$. Then $f_i^* \alpha$ is represented by the cellular cochain β_i that sends each 1-cell σ to the winding number of $f_i|_{\sigma}$ around S^1 . The assumption $f_0^* \alpha = f_1^* \alpha$ then means that there is a cellular 0-cochain $\eta \in C^0(X; \mathbb{Z})$ with $\beta_0 - \beta_1 = \delta \eta$. That is, if σ is a 1-cell with vertices $\sigma(0)$ and $\sigma(1)$, then

$$\beta_0(\sigma) - \beta_1(\sigma) = \eta(\sigma(1)) - \eta(\sigma(0)). \quad (10)$$

We now regard f_0 and f_1 as defining a map $X \times \{0, 1\} \rightarrow S^1$, and we want to extend this to a homotopy $X \times [0, 1] \rightarrow S^1$. We extend over $X^0 \times [0, 1]$ such that if x is a 0-cell, then the restriction to $\{x\} \times [0, 1]$ has winding number $-\eta(x)$ around S^1 . Then equation (10) implies that there is no obstruction to extending the homotopy over $X^1 \times [0, 1]$. Finally, for $k \geq 2$, if the homotopy has been extended over $X^{k-1} \times [0, 1]$, then there is no obstruction to extending the homotopy over $X^k \times [0, 1]$, since $\pi_k(S^1) = 0$. \square

Remark 1.46 The group structure on S^1 induces a group structure on $[X, S^1]$, and it is easy to see that this agrees with the group structure on $H^1(X; \mathbb{Z})$ under the above bijection.

This theorem can be regarded as giving a geometric interpretation of $H^1(X; \mathbb{Z})$, and this can be generalized in various directions. To start, consider

¹⁸We can do this because in general, any CW pair (X, A) has the homotopy extension property. That is, any map $(X \times \{0\} \cup (A \times I)) \rightarrow Y$ extends to a map $X \times I \rightarrow Y$. One constructs the extension cell-by-cell, similarly to arguments we have given previously. In the present case, A is the 0-skeleton of X . This argument generalizes to prove the *cellular approximation theorem*, which asserts that any continuous map between CW complexes is homotopic to a cellular map.

a path connected space Y with only one nontrivial homotopy group, i.e.

$$\pi_k(Y) \simeq \begin{cases} G, & k = n, \\ 0, & k \neq n. \end{cases}$$

Such a space is called an *Eilenberg-MacLane space*. For any G and n , as long as G is abelian when $n > 1$, there exists a CW complex Y with this property¹⁹. We will show later that the CW complex Y is unique up to homotopy equivalence. We denote such a CW complex Y by $K(G, n)$. For example, $K(\mathbb{Z}, 1) = S^1$, and we will see later that $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

Theorem 1.47 *If G is abelian, then for any CW complex X , there is a natural isomorphism*

$$H^n(X; G) \simeq [X, K(G, n)].$$

The proof of this theorem is a straightforward generalization of the previous proof. By the Hurewicz theorem and the universal coefficient theorem, an identification of $\pi_n(K(G, n))$ with G induces an identification

$$H^n(K(G, n), G) = \text{Hom}(G, G).$$

So there is a canonical element $\text{id}_G \in H^n(K(G, n), G)$. The map $[X, K(G, n)] \rightarrow H^n(X; G)$ then sends $[f] \mapsto f^*(\text{id}_G)$. In proving the surjectivity of this map, when extending over the $(n + 1)$ -skeleton, we need the following “homotopy addition lemma”, which generalizes Lemma 1.6(b).

Lemma 1.48 *Let X be a CW complex and let $f : X^k \rightarrow Y$. Suppose that $f(X^{k-1}) = \{y_0\}$ for some base point $y_0 \in Y$. Suppose also that $k \geq 2$ or that $\pi_1(Y, y_0)$ is abelian. Then for any $(k + 1)$ -cell $e : D^{k+1} \rightarrow X$,*

$$\sum_{\sigma} \langle \partial e, \sigma \rangle [f]_{\sigma} = [f \circ e|_{S^k}] \in \pi_k(Y, y_0).$$

¹⁹To construct Y , consider a presentation of G by generators and relations. Let Y^n be a wedge of n -spheres, one for each generator of G . For each relation, attach an $(n + 1)$ -cell whose boundary is the sum of the n -spheres in the relation. Now we have a CW complex Y^{n+1} with $\pi_n(Y^{n+1}) \simeq G$, and $\pi_i(Y^{n+1}) = 0$ for $i < n$. However $\pi_{n+1}(Y^{n+1})$ might not be zero. One can attach $(n + 2)$ -cells to kill the generators of $\pi_{n+1}(Y^{n+1})$ (because the cellular approximation theorem implies that a generator of $\pi_{n+1}(Y^{n+1})$ can be represented by a cellular map on $S^{n+1} \rightarrow Y^{n+1}$), and one can check that this does not mess up π_i for $i \leq n$. One then attaches $(n + 3)$ -cells to kill π_{n+2} , and so on. The resulting CW complex might be very complicated.

Proof. Since f descends to X/X^{k-1} , we can assume without loss of generality that X^{k-1} is a point x_0 . Suppose now that $k \geq 2$. Then by the Hurewicz theorem,

$$\pi_k(X^k, x_0) = H_k(X^k) = C_k(X). \quad (11)$$

By equation (11) and the naturality of the Hurewicz isomorphism under the map $e|_{S^k} : S^k \rightarrow X^k$, we have

$$e|_{S^k} = \sum_{\sigma} \langle \partial e, \sigma \rangle \sigma \in \pi_k(X^k, x_0).$$

Applying f_* to this equation completes the proof.

The case when $k = 1$ and $\pi_1(Y, y_0)$ is abelian follows a similar argument, using the abelianization of $\pi_1(X^1, x_0)$ in equation (11). \square

1.8 Orientations of sphere bundles

Definition 1.49 Let E be an S^k -bundle over B with $k \geq 0$. An *orientation* of E is a choice of generator of the reduced homology $\tilde{H}_k(E_x) \simeq \mathbb{Z}$ for each $x \in B$. This should depend continuously on x in the following sense. If E is trivial over a subset $U \subset B$, then a trivialization $E|_U \simeq U \times S^k$ determines identifications $E_x \simeq S^k$, and hence isomorphisms $\tilde{H}_k(E|_x) \simeq \tilde{H}_k(S^k)$, for each $x \in U$. We require that if U is connected, then the chosen generators of $\tilde{H}_k(E|_x)$ all correspond to the same generator of $\tilde{H}_k(S^k)$. Note that this continuity condition does not depend on the choice of trivialization over U .

The bundle E is *orientable* if it possesses an orientation. It is *oriented* if moreover an orientation has been chosen.

Example 1.50 If B is a smooth n -dimensional manifold, then choosing a metric on B gives rise to an S^{n-1} -bundle $STB \rightarrow B$ consisting of unit vectors in the tangent bundle, and an orientation of STB is the same as an orientation of B in the usual sense.

Example 1.51 The mapping torus of a homeomorphism $f : S^k \rightarrow S^k$, regarded as an S^k -bundle, is orientable if and only if f is orientation-preserving.

Exercise 1.52 Oriented S^1 -bundles over S^2 , up to orientation-preserving isomorphism, are classified by \mathbb{Z} . (We will prove a generalization of this in Theorem 1.63 below.)

Another way to define orientations is as follows. If E is an S^k -bundle, define the *orientation bundle* $\mathcal{O}(E)$ to be the set of pairs (x, \mathfrak{o}_x) where $x \in B$ and \mathfrak{o}_x is a generator of $\tilde{H}_k(E_x)$. We topologize this as follows. If U is an open subset of B and \mathfrak{o} is an orientation of $E|_U$, let $V(U, \mathfrak{o}) \subset \mathcal{O}(E)$ denote the set of pairs (x, \mathfrak{o}_x) where $x \in U$ and \mathfrak{o}_x is the generator of $\tilde{H}_k(E_x)$ determined by \mathfrak{o} . The sets $V(U, \mathfrak{o})$ are a basis for a topology on $\mathcal{O}(E)$. With this topology, $\mathcal{O}(E) \rightarrow B$ is a 2:1 covering space. It is easy to see that an orientation of E is equivalent to a section of $\mathcal{O}(E)$.

We now want to give a criterion for orientability of a sphere bundle. By covering space theory, a path $\gamma : [0, 1] \rightarrow B$ induces a bijection²⁰

$$\Phi_\gamma : \mathcal{O}(E)_{\gamma(0)} \longrightarrow \mathcal{O}(E)_{\gamma(1)}.$$

Moreover, if γ is homotopic to γ' rel endpoints, then $\Phi_\gamma = \Phi_{\gamma'}$. So if $x_0 \in B$ is a base point, we obtain a “monodromy” homomorphism

$$\Phi : \pi_1(B, x_0) \rightarrow \text{Aut}(\mathcal{O}(E)_{x_0}) = \mathbb{Z}/2. \quad (12)$$

Since $\mathbb{Z}/2$ is abelian, this homomorphism descends to the abelianization of $\pi_1(B, x_0)$, so combining these homomorphisms for all path components of B gives a map $H_1(B) \rightarrow \mathbb{Z}/2$. By the universal coefficient theorem, this is equivalent to an element of $H^1(B; \mathbb{Z}/2)$, which we denote by

$$w_1(E) \in H^1(B; \mathbb{Z}/2).$$

Proposition 1.53 *Let $E \rightarrow B$ be an S^k -bundle with $k \geq 1$. Assume that the path components of B are connected (e.g. B is a CW complex). Then:*

- (a) E is orientable if and only if $w_1(E) = 0 \in H^1(B; \mathbb{Z}/2)$.
- (b) If E is orientable, then the set of orientations of E is an affine space²¹ over $H^0(B; \mathbb{Z}/2)$.

²⁰Specifically, for each $t \in [0, 1]$, the inclusion $(\gamma^*E)_t \rightarrow \gamma^*E$ is a homotopy equivalence and hence induces an isomorphism on \tilde{H}_k .

²¹An *affine space* over an abelian group G is a set X with a free and transitive G -action. For $x, y \in X$, we can define the difference $x - y \in G$ to be the unique $g \in G$ such that $g \cdot y = x$. The choice of an “origin” $x_0 \in X$ determines a bijection $G \rightarrow X$ via $g \mapsto g \cdot x_0$. However this identification depends on the choice of x_0 . For example, if A is an $m \times n$ real matrix and $\mathbf{b} \in \mathbb{R}^m$, then the set $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$, if nonempty, is naturally an affine space over \mathbb{R}^k for some k .

Proof. (a) Without loss of generality, B is path connected. Since $\mathcal{O}(E) \rightarrow B$ is a 2:1 covering space, it has a section if and only if it is trivial. By elementary covering space theory, this covering space is trivial if and only if the monodromy (12) is trivial.

(b) $H^0(B; \mathbb{Z}/2)$, regarded as the set of maps $B \rightarrow \mathbb{Z}/2$ that are constant on each path component, acts on the set of orientations of E in an obvious manner. This action is clearly free, and the continuity condition for orientations implies that it is transitive. \square

When B is a CW complex, we can understand (a) in terms of obstruction theory as follows. We can arbitrarily choose an orientation over the 0-skeleton. The obstruction to extending this over the 1-skeleton is a 1-cocycle $\alpha \in C^1(B; \mathbb{Z}/2)$. This 1-cycle represents the class $w_1(E)$. If $\alpha = d\beta$, then β tells us how to switch the orientations over the 0-skeleton so that they extend over the 1-skeleton. There is then no further obstruction to extending over the higher skeleta.

Exercise 1.54 w_1 is natural, in that if $f : B' \rightarrow B$, then

$$w_1(f^*E) = f^*w_1(E) \in H^1(B; \mathbb{Z}/2). \quad (13)$$

Equation (13) implies that w_1 is what is called a “characteristic class” of sphere bundles. We will study characteristic classes more systematically later. Our next example of a characteristic class is the Euler class.

1.9 The Euler class of an oriented sphere bundle

Let $E \rightarrow B$ be an oriented S^k bundle with $k \geq 1$. Assume temporarily that B is a CW complex. We now define the *Euler class*

$$e(E) \in H^{k+1}(B; \mathbb{Z}). \quad (14)$$

This is the “primary obstruction” to the existence of a section of E , in the sense of Proposition 1.58 below.

First, we choose a section s_0 of E over the 0-skeleton B^0 (just pick any point in the fiber over each 0-cell). Now if we have a section s_{i-1} over the $(i-1)$ -skeleton, and if $i \leq k$, then we can extend to a section s_i over the i -skeleton. The reason is that if $e : D^i \rightarrow B$ is an i -cell, then we know

that the bundle²² e^*E over D^i is trivial, so we can identify $e^*E \simeq D^i \times S^k$. The section s_{i-1} induces a map $\partial D_i = S^{i-1} \rightarrow S^k$, and this extends over D_i because $\pi_{i-1}(S^k) = 0$. By induction we obtain a section s_k over the k -skeleton.

Now let us try to extend s_k over the $(k+1)$ -skeleton. Let $e : D^{k+1} \rightarrow B$ be a $(k+1)$ -cell. We choose an *orientation-preserving* trivialization $e^*E \simeq D^{k+1} \times S^k$. Then s_k defines a map $\partial D^{k+1} \rightarrow S^k$, which we can identify with an element of $\pi_k(S^k) = \mathbb{Z}$. This assignment of an integer to each $(k+1)$ -cell defines a cochain

$$\mathfrak{o}(s_k) \in C^{k+1}(B; \mathbb{Z}).$$

Lemma 1.55 $\mathfrak{o}(s_k)$ is a cocycle: $\delta\mathfrak{o}(s_k) = 0$.

*Proof.*²³ Consider a $(k+2)$ -cell $\xi : D^{k+2} \rightarrow B$. We need to show that

$$\sum_{\sigma} \langle \partial\xi, \sigma \rangle \mathfrak{o}(s_k)(\sigma) = 0. \quad (15)$$

Let us choose an identification of the interior of each $(k+1)$ -cell in B with the interior of D^{k+1} . In the interior of each $(k+1)$ -cell σ , let p_{σ} denote the center point of σ .

By a bit of smooth topology, we can homotope the attaching map $\xi|_{S^{k+1}}$ so that it is transverse to each p_{σ} . Then²⁴

- (i) The inverse image of p_{σ} is a finite set of points $x \in S^{k+1}$, to each of which is associated a sign $\epsilon(x)$.
- (ii) The inverse image of a small $(k+1)$ -disk D_{σ} around p_{σ} consists of one $(k+1)$ -disk $D_x \subset S^{k+1}$ for each $x \in \xi^{-1}(p_{\sigma})$, such that the restriction of ξ to D_x is a homeomorphism which is orientation-preserving iff $\epsilon(x) = +1$.

²²The reason that we use pullback bundles here instead of restriction is that the map e might not be injective on ∂D^i .

²³I got stuck on this in class. It's a simple idea, but a bit hard to explain. Try drawing pictures in the case $k = 1$.

²⁴One can also obtain these conclusions without using smooth topology, see e.g. Hatcher. This homotopy may change B , but it does not change the left side of the equation (15) that we are trying to prove.

(iii)

$$\langle \partial\xi, \sigma \rangle = \sum_{x \in \xi^{-1}(p_\sigma)} \epsilon(x).$$

Next, we can extend the section s_k over $B^{k+1} \setminus \bigcup_\sigma p_\sigma$. Also, choose a trivialization $\xi^*E \simeq D^{k+2} \times S^k$. The extended section s_k then defines a map $f : S^{k+1} \setminus \bigcup_\sigma \xi^{-1}(p_\sigma) \rightarrow S^k$. By (ii) above, for each $x \in \xi^{-1}(p_\sigma)$, we have

$$\epsilon(x)[f|_{\partial D_x}] = \mathfrak{o}(s_k)(\sigma) \in \pi_k(S^k).$$

Combining this with (iii) gives

$$\sum_\sigma \langle \partial\xi, \sigma \rangle \mathfrak{o}(s_k)(\sigma) = \sum_\sigma \sum_{x \in \xi^{-1}(p_\sigma)} [f|_{\partial D_x}] \in \pi_k(S^k).$$

Now let X denote the complement in S^{k+1} of the interiors of the balls D_x ; this is a compact manifold with boundary. By the Hurewicz isomorphism, the right hand side of the above equation can be identified with $-f_*[\partial X] \in H_k(S^k)$. But this is zero, since f extends over X . \square

Definition 1.56 The Euler class (14) is the cohomology class of the cocycle $\mathfrak{o}(s_k)$.

Lemma 1.57 *The cohomology class $e(E)$ is well-defined.*

Proof. Let s_k and t_k be two sections over the k -skeleton. We will construct a cellular cochain $\eta \in C^k(B; \mathbb{Z})$ with

$$\delta\eta = \mathfrak{o}(s_k) - \mathfrak{o}(t_k). \tag{16}$$

The idea is to try to find a homotopy from s_k to t_k . Consider the pullback of E to $B \times I$ with the product CW structure; we can then regard s_k and t_k as sections defined over $B^k \times \{0\}$ and $B^k \times \{1\}$ respectively. There is no obstruction to extending this over $B^{k-1} \times I$, so that we now have a section u_k over the k -skeleton of $B \times I$. By Lemma 1.55,

$$\delta\mathfrak{o}(u_k) = 0 \in C^{k+2}(B \times I; \mathbb{Z}). \tag{17}$$

Now define the required cellular cochain $\eta \in C^k(B; \mathbb{Z})$ as follows: if ρ is a k -cell in B , then

$$\eta(\rho) := \mathfrak{o}(u_k)(\rho \times I) \in \mathbb{Z}$$

is the obstruction to obstruction to extending the homotopy over ρ . Then equation (16), evaluated on a $(k+1)$ -cell σ in B , follows from equation (17) evaluated on the $(k+2)$ -cell $\sigma \times I$ in $B \times I$. \square

To put this all together, we have:

Proposition 1.58 *Let E be an oriented S^k -bundle over a CW complex B . Then there exists a section of E over the $(k+1)$ -skeleton B^k if and only if $e(E) = 0 \in H^k(B; \mathbb{Z})$.*

Proof. (\Rightarrow) If there exists a section s_k over the k -skeleton that extends over the $(k+1)$ -skeleton, then by construction the cocycle $\mathfrak{o}(s_k) = 0$.

(\Leftarrow) Suppose that $e(E) = 0$. Let s_k be a section over the k -skeleton; then we know that $\mathfrak{o}(s_k) = \delta\eta$ for some $\eta \in C^k(B; \mathbb{Z})$. Keeping s_k fixed over the $(k-1)$ -skeleton, we can modify it to a new section t_k over the k -skeleton, so that over each k -cell, s_k and t_k differ by η ; see Exercise 1.59 below. Then by equation (16), $\mathfrak{o}(t_k) = 0$, which means that t_k extends over the $(k+1)$ -skeleton. \square

Exercise 1.59 Let X be a path connected space such that $\pi_k(X)$ is abelian and the action of π_1 on π_k is trivial (so that $\pi_k(X)$ is just the set of homotopy classes of maps $S^k \rightarrow X$ with no base point). Let $f : S^{k-1} \rightarrow X$. Then the set of homotopy classes of extensions of f to a map $D^k \rightarrow X$ is an affine space over $\pi_k(X)$. The difference between two extensions g_+ and g_- is obtained by regarding g_+ and g_- as maps defined on the northern and southern hemispheres respectively of S^k , and gluing them together along the equator (where they are both equal to f) to obtain a map $S^k \rightarrow X$.

We will show later that the Euler class is natural and does not depend on the CW structure on B . Indeed there are alternate definitions which do not use a CW structure.

Example 1.60 For an oriented S^1 -bundle over S^2 , the Euler class agrees with the rotation number. This is easy to see if we use a cell decomposition of S^2 with two 2-cells corresponding to the northern and southern hemispheres.

Remark 1.61 The Euler class is called the “primary obstruction” to the existence of a section. There can also be “secondary obstructions” and higher obstructions, involving higher homotopy groups of spheres. That is, when

$\dim(B) > k + 1$, it is possible that $e(E) = 0$ and yet no section over all of B exists. An example of this (which requires a bit of proof) is given by the S^2 -bundle over S^4 in which a nonzero element of $\pi_3(SO(3)) = \mathbb{Z}$ is used in the clutching construction.

On the other hand, we do have:

Proposition 1.62 *An oriented S^1 -bundle over a CW complex has a section if and only if its Euler class vanishes.*

Proof. If the Euler class vanishes, then there is a section over the 2-skeleton, and since the higher homotopy groups of S^1 are trivial, there is no obstruction to extending this section over all higher skeleta. \square

The Euler class can also be used to help classify sphere bundles. Clearly, if two oriented sphere bundles are isomorphic via an orientation-preserving isomorphism, then they have the same Euler class. In favorable cases, the converse is true. In particular, we have the following nice higher-dimensional analogue²⁵ of Theorem 1.45.

Theorem 1.63 *Let B be a CW complex. Then the Euler class defines a bijection from the set of oriented S^1 bundles over B , up to orientation-preserving isomorphism, to $H^2(B; \mathbb{Z})$.*

Proof. The proof that the Euler class is injective is similar to the proof of Proposition 1.62, using the fact that the space of orientation-preserving homeomorphisms of S^1 deformation retracts onto S^1 . We leave the details as an exercise.

Now let us prove that the Euler class is surjective. Let $\mathfrak{o} \in C^2(B; \mathbb{Z})$ be a cellular cocycle. We will construct an oriented S^1 bundle $E \rightarrow B$ and a section s_1 of E over the 1-skeleton such that \mathfrak{o} is the obstruction to extending s_1 over the 2-skeleton.

Over the 1-skeleton, we define $E|_{B^1} = B^1 \times S^1$, and let s_1 be a constant section.

We now inductively extend E over the k -skeleton by gluing in one copy of $D^k \times S^1$ for each k -cell $\sigma : D^k \rightarrow E$. To do so we need a gluing map $S^{k-1} \times S^1 \rightarrow E|_{B^{k-1}}$ which projects to the attaching map $\sigma|_{S^{k-1}} : S^{k-1} \rightarrow B^{k-1}$ and which restricts to an orientation-preserving homeomorphism on

²⁵One can further identify $H^3(B; \mathbb{Z})$ with isomorphism classes of “gerbes” on B . But that’s beyond the scope of this course.

each fiber. That is, we need to specify an orientation-preserving bundle isomorphism

$$S^{k-1} \times S^1 \longrightarrow (\sigma|_{S^{k-1}})^*(E|_{B^{k-1}}). \quad (18)$$

When $k = 2$, the map (18) is an orientation-preserving bundle isomorphism $S^1 \times S^1 \rightarrow S^1 \times S^1$, i.e. a map $S^1 \rightarrow \text{Homeo}^+(S^1)$, where the superscript ‘+’ indicates orientation-preserving. We choose this to be a map $S^1 \rightarrow S^1$ with degree $\mathfrak{o}(\sigma) \in \mathbb{Z}$. This ensures that \mathfrak{o} is the obstruction to extending s_1 over the 2-skeleton.

When $k \geq 3$, we can choose any bundle isomorphism (18); we just need to show that $(\sigma|_{S^{k-1}})^*(E|_{B^{k-1}})$ is trivial. That is, we need to check that the Euler class

$$e((\sigma|_{S^{k-1}})^*(E|_{B^{k-1}})) = 0 \in H^2(S^{k-1}; \mathbb{Z}).$$

When $k = 3$, this follows from the cocycle condition $\delta\mathfrak{o} = 0$. When $k > 3$, this is automatic. \square

1.10 Homology with twisted coefficients

A natural context in which to generalize some of the previous discussion is provided by homology with “twisted” or “local” coefficients.

Definition 1.64 A *local coefficient system* on a space X consists of the following:

- (a) For each $x \in X$, an abelian group G_x ,
- (b) for each path $\gamma : [0, 1] \rightarrow X$, a homomorphism (necessarily an isomorphism) $\Phi_\gamma : G_{\gamma(0)} \xrightarrow{\cong} G_{\gamma(1)}$, such that:
 - (i) Φ_γ depends only on the homotopy class of γ rel endpoints,
 - (ii) If γ_1 and γ_2 are composable paths then $\Phi_{\gamma_1\gamma_2} = \Phi_{\gamma_1}\Phi_{\gamma_2}$,
 - (iii) If γ is a constant path then Φ_γ is the identity.

We sometimes denote a local coefficient system by \mathcal{G} or by $\{G_x\}$, leaving the isomorphisms Φ_γ implicit.

Example 1.65 A *constant* local coefficient system is obtained by setting $G_x = G$ for some fixed group G , and $\Phi_\gamma = \text{id}_G$ for all γ .

Example 1.66 If $n > 1$, or $n = 1$ and $\pi_1(X, x_0)$ is abelian for all $x_0 \in X$, then $\{\pi_n(X, x)\}$ is a local coefficient system on X .

Example 1.67 If $E \rightarrow B$ is a Serre fibration, then $\{H_*(E_x)\}$ is a local coefficient system on B .

Example 1.68 A *bundle of groups* is a covering space $\tilde{X} \rightarrow X$ such that each fiber has the structure of an abelian group, which depends continuously on $x \in X$ (in the sense that there are local trivializations which are group isomorphisms on each fiber). Any bundle of groups gives rise to a local coefficient system on X . If X is a “reasonable” space, then the converse is true.

Remark 1.69 If X is path connected and $x_0 \in X$ is a base point, then a local coefficient system on X determines a monodromy homomorphism

$$\pi_1(X, x_0) \longrightarrow \text{Aut}(G_{x_0}).$$

Conversely, if G is any abelian group, then any homomorphism $\pi_1(X, x_0) \rightarrow \text{Aut}(G)$ is the monodromy of a local coefficient system on X with $G_x = G$.

Now let $\mathcal{G} = \{G_x\}$ be a local coefficient system on X . We want to define the homology with local coefficients $H_*(X, \mathcal{G})$, as well as the cohomology $H^*(X, \mathcal{G})$.

Let $\sigma : I^k \rightarrow X$ be a singular cube. For every path $\gamma : I \rightarrow I^k$, the isomorphism $\Phi_{\sigma \circ \gamma}$ defines an isomorphism $G_{\sigma(\gamma(0))} \simeq G_{\sigma(\gamma(1))}$. Since I^k is contractible, these isomorphisms are canonical, so that the groups $G_{\sigma(t)}$ for $t \in I^k$ are all isomorphic to a single group, which we denote by G_σ . Also, if σ' is a face of σ , then $G_{\sigma'} = G_\sigma$.

Now define $C_*(X, \{G_x\})$ to be the free \mathbb{Z} -module generated by pairs (σ, g) where $g \in G_\sigma$, modulo degenerate cubes as usual, and modulo the relation

$$(\sigma, g_1) + (\sigma, g_2) = (\sigma, g_1 + g_2).$$

Define the differential $\partial(\sigma, g)$ in the usual way, where g comes along for the ride. The homology of this complex is $H_*(X, \{G_x\})$. The cohomology $H^*(X, \{G_x\})$ is the cohomology of the chain complex consisting of functions that assign to each singular cube σ an element of the group G_σ .

If X is a CW complex, then we can analogously define cellular homology and cohomology with local coefficients.

Example 1.70 A local coefficient system \mathcal{G} on S^1 is specified by a group G and a monodromy map $\Phi : G \rightarrow G$. If we choose a cell structure with one 0-cell e_0 and one 1-cell e_1 , then under appropriate identifications, the differential in the cellular chain complex $C_*(S^1; \mathcal{G})$ is given by

$$\partial(e_1, g) = (e_0, g - \Phi g).$$

Hence $H_0(S^1; \mathcal{G}) = G / \text{Im}(1 - \Phi)$ and $H_1(S^1; \mathcal{G}) = \text{Ker}(1 - \Phi)$. Also $H^k(S^1; \mathcal{G}) = H_{1-k}(S^1; \mathcal{G})$.

Example 1.71 Let $E \rightarrow B$ be an S^k -bundle with $k \geq 1$, but without an orientation. Then our previous construction of the Euler class generalizes to give a cohomology class

$$e(E) \in H^{k+1}(B; \{H_k(E_x)\}),$$

which vanishes iff E has a section over the $(k + 1)$ -skeleton.

Example 1.72 Let E be a fiber bundle over a CW-complex B with fiber F . Suppose that F is path connected. Let k be the smallest positive integer such that $\pi_k(F) \neq 0$; suppose that $\pi_k(F)$ is abelian and that $\pi_1(F)$ acts trivially on $\pi_k(F)$. Then the previous example generalizes to give a cohomology class

$$\alpha \in H^{k+1}(B; \{\pi_k(E_x)\})$$

which vanishes iff E has a section over the $(k + 1)$ -skeleton.

Example 1.73 One can use local coefficients to generalize Poincaré duality to manifolds which are not necessarily oriented or even orientable. Any n -dimensional manifold X has a local coefficient system \mathcal{O} , such that $\mathcal{O}_x = H_n(U, U \setminus \{x\}) \simeq \mathbb{Z}$, where U is a Euclidean neighborhood of x . An orientation of X is a section of \mathcal{O} which restricts to a generator of each fiber. In any case, regardless of whether or not an orientation exists, if X is a compact manifold without boundary, and if \mathcal{G} is any local coefficient system on X , then there is a canonical isomorphism

$$H^k(X; \mathcal{G}) = H_{n-k}(X; \mathcal{G} \otimes \mathcal{O}).$$

This isomorphism is given by cap product with a canonical fundamental class $[X] \in H_n(X; \mathcal{O})$. To prove this, take your favorite proof of Poincaré duality and insert local coefficients everywhere.

1.11 Whitehead's theorem

The following theorem shows that homotopy groups give a criterion for a map to be a homotopy equivalence.

Theorem 1.74 *Let X and Y be path connected CW complexes and let $f : X \rightarrow Y$ be a continuous map. Suppose that f induces isomorphisms on all homotopy groups. Then f is a homotopy equivalence.*

Proof. We first reduce to a special case. Define the *mapping cone*

$$C_f := \frac{(X \times I) \cup Y}{(x, 1) \sim f(x)}.$$

Then f is the composition of two maps $X \rightarrow C_f \rightarrow Y$, where the first map is an inclusion sending $x \mapsto (x, 0)$, and the second map sends $(x, t) \mapsto f(x)$ and $y \mapsto y$. Furthermore, the map $C_f \rightarrow Y$ is a homotopy equivalence since it comes from a deformation retraction of C_f onto Y . So it is enough to show that the inclusion $X \rightarrow C_f$ is a homotopy equivalence. By the cellular approximation theorem, to prove Whitehead's theorem we may assume that f is cellular. Then C_f is a CW complex and $X \times \{0\}$ is a subcomplex. In conclusion, to prove Whitehead's theorem, we may assume that X is a subcomplex of Y and f is the inclusion.

So assume that X is a subcomplex of Y and that the inclusion induces isomorphisms on all homotopy groups. We now construct a deformation retraction of Y onto X . This consists of a homotopy $F : Y \times I \rightarrow X$ such that $F(x, t) = x$ for all $x \in X$ and $t \in I$, and $F(y, 1) \in X$ for all $y \in Y$. We construct F one cell at a time. This reduces to the following problem: given

$$\begin{aligned} F : (D^k \times \{0\}) \cup (\partial D^k \times I) &\longrightarrow Y, \\ \partial D^k \times \{1\} &\longmapsto X, \end{aligned} \tag{19}$$

extend F to a map $D^k \times I \rightarrow Y$ sending $D^k \times \{1\} \rightarrow X$.

The restriction of F to $\partial D^k \times \{1\}$ defines an element $\alpha \in \pi_{k-1}(X)$ (mod the action of $\pi_1(X)$). Since F extends to a map (19), it follows that α maps to 0 in $\pi_{k-1}(Y)$. Since the inclusion induces an injection on π_{k-1} , it follows that α extends to a map $D^k \times \{1\} \rightarrow X$. So now F is defined on $\partial(D^k \times I) \simeq S^k$, and we need to extend F to a map to all of Y . This may not be possible. The map we have so far represents an element $\beta \in \pi_k(Y)$ (mod the action of $\pi_1(Y)$), and we need this element to be zero. If $\beta \neq 0$, then since the

map $\pi_k(X) \rightarrow \pi_k(Y)$ is surjective, we can change our choice of extension of F over $D^k \times I$ to arrange that $\beta = 0$. \square

Corollary 1.75 *A path connected CW complex $K(G, n)$ satisfying $\pi_n(K(G, n)) \simeq G$ and $\pi_i(K(G, n)) = 0$ for $i \neq n$ is unique up to homotopy equivalence.*

Proof. Let $K'(G, n)$ be another such CW complex. If we choose identifications of $\pi_n(K(G, n))$ and $\pi_n(K'(G, n))$, then we know that for any CW complex X , we have $[X, K(G, n)] = [X, K'(G, n)] = H^n(X; G)$. In particular, $[K'(G, n), K(G, n)] = \text{Hom}(G, G)$ contains a canonical element id_G corresponding to a homotopy class $f : K'(G, n) \rightarrow K(G, n)$. By the naturality of the Hurewicz isomorphism, f induces an isomorphism on π_n , and hence on all homotopy groups. Therefore f is a homotopy equivalence. \square

In general, it can be hard to check that a map induces isomorphisms on all homotopy groups, since the latter are hard to compute. However, the proof of Whitehead's theorem also shows the following:

Theorem 1.76 *Let X and Y be path connected CW complexes and let $f : X \rightarrow Y$ be a continuous map. Let $n = \max(\dim(X), \dim(Y))$. Suppose that:*

- (a) *For all $i \leq n$, f induces an isomorphism on π_i .*
- (b) *f induces a surjection on π_{n+1} .*

Then f is a homotopy equivalence.

Remark 1.77 If all we know is that X and Y have isomorphic homotopy groups, then X and Y need not be homotopy equivalent (i.e. the isomorphisms on homotopy groups might not be induced by a map $f : X \rightarrow Y$).

2 Characteristic classes

3 Spectral sequences

4 Morse theory