ON BRAID GROUPS AND HOMOTOPY GROUPS

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Abstract. This article is an exposition of certain connections between the braid groups, classical homotopy groups, as well as Lie algebras attached to the descending central series of pure braid groups arising as Vassiliev invariants of pure braids. Natural related questions are posed at the end of this article.

1. Introduction

The purpose of this article is to give an exposition of certain connections between the braid groups [2], classical homotopy groups in joint work of Jon Berrick, Yan-Loi Wong and the authors [5, 1, 18], as well as Lie algebras attached to the descending central series of pure braid groups arising as Vassiliev invariants of the pure braid groups as developed by T. Kohno [11, 12]. Natural related questions are posed at the end of this article .

The main feature of this article is to identify certain "non-standard" free subgroups of the braid groups via Vassiliev invariants of pure braids. The second feature is to indicate natural ways in which these subjects fit together with classical homotopy theory. This article is an attempt to draw together some of these connections.

Although not yet useful for direct computations, there is a strong connection between braid groups and homotopy groups. The braids which naturally arise in this setting also give a large class of special knots arising from Brunnian braids as described below as well as in [17]. It is natural to wonder whether and how these fit together.

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Contents

2. Braid groups, Vassiliev invariants of pure braids and certain free subgroups of braid groups

The section addresses a naive construction with the braid groups arising as a "cabling" construction. This construction is interpreted in later sections in terms of the structure of braid groups, Vassiliev invariants of pure braids as developed by Toshitake Kohno [11, 12], associated Lie algebras and the homotopy groups of the 2-sphere [5, 1, 18].

Let B_k denote Artin's k-stranded braid group while P_k denotes the pure k-stranded braid group, the subgroup of B_k which corresponds to the trivial permutation of the endpoints of the strands. The group P_k is the fundamental group of the configuration space of ordered k-tuples of distinct points in the plane

$$Conf(\mathbb{R}^2, k)$$

for which $Conf(M, k) = \{(m_1, m_2, \cdots, m_k) | m_i \neq m_j \text{ for all } i \neq j\}$ for any space M.

The group B_k is the fundamental group of the orbit space

$$Conf(\mathbb{R}^2, k)/\Sigma_k$$

obtained from the natural, free (left)-action of the symmetric group on k letters Σ_k. The k-stranded braid group of an arbitrary connected surface S, $B_k(S)$, is defined to be the fundamental group of the configuration space of unordered k-tuples of distinct points in S, $\pi_1 Conf(S, k)/\Sigma_k$. The pure braid group $P_k(S)$ is defined to be the fundamental group $\pi_1 Conf(S, k)$.

The pure braid groups P_k will be seen to assemble into a topological space which "contains" the loop space of the 2-sphere in a natural way. To address this last point, first consider the free group on N letters $F_N = F_N[y_1, \cdots, y_N]$ together with elements x_i for $1 \leq i \leq N$ in P_{N+1} given by the naive "cabling" pictured in Figure 2.1 below.

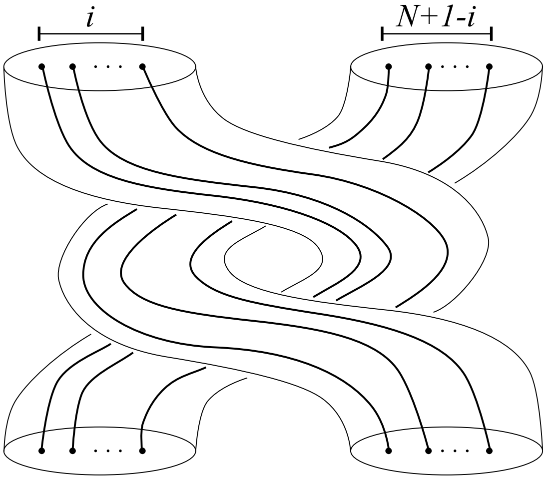

Figure 2.1. The braid x_i in P_{N+1}.

The braid x_1 with $N = 1 = i$ in Figure 2.1 is Artin's generator $A_{1,2}$ of P_2. The braids x_i for $1 \leq i \leq N$ in Figure 2.1 yield homomorphisms from a free group on N letters $F_N = F_N[y_1, \cdots, y_N]$ to P_{N+1}

$$\Theta_N \colon F_N[y_1, \cdots, y_N] \to P_{N+1}$$

defined on generators y_i in F_N by the formula

$$\Theta_N(y_i) = x_i.$$

The maps Θ_N are the subject of [5] where it is shown $\Theta_N\colon F_N \to P_{N+1}$ is faithful for every N. Three natural questions arise: (1) Why would one want to know whether Θ_n is faithful, (2) are there sensible applications and (3) why is Θ_n faithful ? The answers to these three questions provide the main content of this expository article.

3. On Θ_n

This section addresses one reason why the map Θ_n is faithful [5]. The method of proof is to appeal to the structure of the Lie algebras obtained from the descending central series for both the source and the target of Θ_n. The associated graded Lie algebra obtained from the descending central series for the target yields Vassiliev invariants of pure braids by work of Kohno [11, 12]. This Lie algebra has been used by both Kohno and Drinfel'd [7] in their work on the KZ equations. The Lie algebra obtained from the descending central series of the free group F_N is a free Lie algebra by a classical result due to P. Hall [10, 15].

The proof yields more information than just the fact that Θ_N is faithful. The method of proof gives a natural connection of Vassiliev invariants of braids to a classical spectral sequence abutting to the homotopy groups of the 2-sphere. Section 8 below provides an elucidation of this interconnection.

Recall that a discrete group Γ is said to be residually nilpotent provided

$$\cap_{i\geq 1}\Gamma^i(\pi) = \{identity\}$$

where $\Gamma^i(\pi)$ denotes the i-th stage of the descending central series for π. Examples of residually nilpotent groups are free groups, and P_n. Next recall the Lie algebra obtained as the associated graded of the descending central series of a discrete group π with $gr_*(\pi)$ given by $gr_i(\pi) = \Gamma^i(\pi)/\Gamma^{i+1}(\pi)$.

Lemma 3.1. (1) *Assume that π is a residually nilpotent group. Let*

$$\alpha\colon \pi \to G$$

be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras

$$gr_*(\alpha)\colon gr_*(\pi) \to gr_*(G)$$

is a monomorphism. Then α is a monomorphism.

(2) *If π is a free group, and $gr_*(\alpha)$ is a monomorphism, then α is a monomorphism.*

Thus one step is to describe the map $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ on the level of associated graded Lie algebras

$$gr_*(\Theta_n)\colon gr_*(F[y_1, y_2, \cdots, y_n]) \to gr_*(P_{n+1}).$$

Recall Artin's generators $A_{i,j}$ for P_{n+1} together with the projections of the $A_{i,j}$ to $gr_*(P_{n+1})$ labeled $B_{i,j}$ [5].

Theorem 3.2. *The induced morphism of Lie algebras*

$$gr_*(\Theta_n)\colon gr_*(F[y_1, y_2, \cdots, y_n]) \to gr_*(P_{n+1})$$

satisfies the formula

$$gr_*(\Theta_n)(y_q) = \Sigma_{1\leq i\leq n-q+1<j\leq n+1} B_{i,j}.$$

To determine the map of Lie algebras more globally, the structure of $gr_*(P_n)$ is useful. The structure of that Lie algebra is given as follows. Let $L[S]$ denote the free Lie algebra over $\mathbb{Z}$ generated by a set S. The next theorem was given in work of Kohno [11, 12] and Falk-Randell [9].

Theorem 3.3. *The Lie algebra* $gr_*(P_n)$ *is the quotient of the free Lie algebra generated by* $B_{i,j}$ *for* $1 \leq i < j \leq n$ *modulo the infinitesimal braid relations (also called the horizontal 4T relations or Yang-Baxter-Lie relations)*

$$gr_*(P_n) = L[B_{i,j} |\, 1 \leq i < j \leq n]/I$$

where I *denotes the 2-sided (Lie) ideal generated by the infinitesimal braid relations as listed next:*

1. $[B_{i,j}, B_{s,t}] = 0$, *if* $\{i, j\} \cap \{s, t\} = \emptyset$.
2. $[B_{i,j}, B_{i,s} + B_{s,j}] = 0$.
3. $[B_{i,j}, B_{i,t} + B_{j,t}] = 0$.

A computation elucidated in section 8 with these maps over the integers gives the following result of [5].

Theorem 3.4. *The maps* $\Theta_n \colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ *on the level of associated graded Lie algebras*

$$gr_*(\Theta_n) \colon gr_*(F[y_1, y_2, \cdots, y_n]) \to gr_*(P_{n+1})$$

are monomorphisms. Thus the maps Θ_n *are monomorphisms.*

Remark: The combinatorial behavior of the map $gr_*(\Theta_n)$ is intricate even though the definition is elementary as well as natural. For example, various powers of 2 arise in the computation of the map

$$gr_*(\Theta_n) \colon gr_*(F[y_1, y_2, \cdots, y_n]) \to gr_*(P_{n+1})$$

for $n > 2$. One example is listed next.

Example 3.5. $\Theta_3([[[x_1, x_2]x_3]x_2]) = -[[[\gamma_1, \gamma_2]\gamma_3]\gamma_2] + 2[[[\gamma_1, \gamma_3]\gamma_2]\gamma_2] + \Delta$ where Δ is independent of the other terms with $\gamma_1 = B_{1,4} + B_{2,4} + B_{3,4}$, $\gamma_2 = B_{3,4}$ and $\gamma_3 = B_{2,4} + B_{3,4}$.

The crucial feature which make the computations effective is the "infinitesimal braid relations". In addition, the behavior of the map $gr_*(\Theta_n)$ is more regular after restricting to certain sub-Lie algebras arising in the third stage of the descending central series [5]. Finally, the maps Θ_n also induce monomorphisms of restricted Lie algebras on passage to the Lie algebras obtained from the mod-p descending central series [5].

4. THE PURE BRAID GROUPS AS A SIMPLICIAL GROUP

Consider the pure braid groups $P_n(S)$ of a surface S with $P_n = P_n(\mathbb{R}^2)$ together with the following natural homomorphisms.

1. There are $n + 1$ homomorphisms

   $$d_i \colon P_{n+1}(S) \to P_n(S)$$

   for $0 \leq i \leq n$ obtained by deleting the $(i+1)$-st strand in $P_{n+1}(S)$. The homomorphisms d_i are induced on the level of fundamental groups of configuration spaces by the projection maps

   $$p_{i+1} \colon Conf(S, n + 1) \to Conf(S, n)$$

   given be deleting the $(i + 1)$-st coordinate.
2. In case $S = \mathbb{R}^2$, there are $n + 1$ homomorphisms

   $$s_i \colon P_{n+1} \to P_{n+2}$$

   obtained by "doubling" the $(i + 1)$-st strand. The homomorphisms s_i are induced on the level of fundamental groups by the maps for configuration spaces

   $$\mathbb{S}_i \colon Conf(\mathbb{R}^2, n + 1) \to Conf(\mathbb{R}^2, n + 2)$$

defined by the formula

$$\mathbb{S}_i(x_1,\cdots,x_{n+1}) = (x_1,\cdots,x_{i+1},\lambda(x_{i+1}),x_{i+2},\cdots,x_{n+1})$$

where $\lambda(x_{i+1}) = x_{i+1} + (\epsilon, 0)$ for $(\epsilon, 0)$ a point in $\mathbb{R}^2$ with

$$\epsilon = (1/2)\cdot min_{t\neq i+1}||x_{i+1} - x_t||.$$

Observe that these homomorphisms satisfy the following three identities:

$$d_id_j = d_{j-1}d_i \text{ if } i < j,$$

$$s_is_j = s_{j+1}s_i \text{ if } i \leq j \text{ and}$$

$$d_is_j = \begin{cases} s_{j-1}d_i & \text{if } i < j, \\ \mathit{identity} & \text{if } i = j \text{ or } i = j+1, \\ s_jd_{i-1} & \text{if } i > j+1. \end{cases}$$

These identities are known as the simplicial identities.

The pure groups thus provide a basic example of a structure known as a simplicial group. That is a collection of groups

$$\{\Gamma_0, \Gamma_1, \Gamma_2, \cdots\}$$

denoted Γ_* with homomorphisms called face operations

$$d_i \colon \Gamma_n \to \Gamma_{n-1},$$

$0 \leq i \leq n$ and homomorphisms called degeneracy operations

$$s_j \colon \Gamma_n \to \Gamma_{n+1},$$

$0 \leq j \leq n$, which satisfy the simplicial identities. The basic example of a simplicial group given by setting

$$\Gamma_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \cdots$ will be denoted AP_* below.

A second example of a simplicial group arises from the cabling operation described in section 2 which yields the homomorphism $\Theta_n \colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$. There is a simplicial group denoted $F[S^1]$ invented by Milnor [13] which in degree n is given by $F[y_1, y_2, \cdots, y_n]$. In addition, the geometric realization of $F[S^1]$ is homotopy equivalent to the loop space of of the 2-sphere ΩS^2[13].

Furthermore, the maps Θ_n yield a morphism of simplicial groups

$$\Theta \colon F[S^1] \to \mathrm{AP}_*$$

which is the subject of section 8 here. Thus, the picture given in Figure 2.1 can be regarded as that of a group theoretic model for the loop space of the 2-sphere in the world of simplicial groups.

5. Pure braid groups of surfaces as Δ-groups

A natural variation of a simplicial group is a Δ-group, a collection of groups

$$\Delta_* = \{\Delta_0, \Delta_1, \cdots\}$$

together with homomorphisms $d_i \colon \Delta_n \to \Delta_{n-1}$, $n \geq 1$, which (1) satisfy the standard commutation formulas for face operations in a simplicial group, but (2) are not necessarily equipped with degeneracy operations. That is

$$d_id_j = d_{j-1}d_i \text{ if } i < j.$$

Examples of Δ-groups are obtained from any simplicial group by retaining the face operations d_i, but omitting the degeneracies.

More examples are obtained from pure braid groups of surfaces. Let S denote a path-connected surface. One basic example in [1] of a Δ-group is $\Delta_*(S)$ defined by

$$\Delta_n(S) = P_{n+1}(S).$$

In case $S = \mathbb{CP}^1 = S^2$, the associated Δ-group is used in [1]. In case S_g is a closed orientable surface, the Δ-group $\Delta_*(S_g)$ does not admit the structure of a simplicial group.

6. Brunnian braids and "almost Brunnian" braids

Consider the k-stranded braid group for any (connected) surface S which is the fundamental group of $Conf(S,k)/\Sigma_k$. Next consider the group of Brunnian braids $Brun_k(S)$ those braids which become trivial after deleting any single strand. Evidently, these braids are a subgroup of the pure braid group $P_k(S)$ since a braid which represents a non-trivial permutation fails to be Brunnian. Observe that the group $Brun_k(S)$ is the intersection of the kernels induced by the maps $p_i\colon Conf(S,k) \to Conf(S,k-1)$ on the level of fundamental groups.

It is convenient to consider a slight variation of Brunnian braids. A group analogous to the group of Brunnian braids is the "almost Brunnian" $(k+2)$-stranded braid group

$$QBrun_{k+2}(S) = \cap_{1\leq i\leq k+1} ker(d_i\colon P_{k+2}(S) \to P_{k+1}(S)).$$

The subgroup $QBrun_{k+2}$ of $P_{k+2}(S)$ consists of those braids which are trivial after deleting any one of the strands $2, 3, \cdots, k+2$, but not necessarily the first. In the case of $S = \mathbb{R}^2$, observe that the map $d_0\colon QBrun_{k+2} \to Brun_{k+1}$ is a split surjection.

The next lemma follows by a direct check of the long exact homotopy sequence obtained from the Fadell-Neuwirth fibrations for configuration spaces [8].

Lemma 6.1. *If $k \geq 3$, then $Brun_k(S)$ and $QBrun_k(S)$ are free groups.*

One question below is to consider the free groups gotten from the intersections $\Theta_k(F_k) \cap Brun_{k+1}$ as well as $\Theta_k(F_k) \cap d_0(QBrun_{k+2})$.

Lemma 6.2. *If $k \geq 3$, then $\Theta_k(F_k) \cap Brun_{k+1}$ as well as $\Theta_k(F_k) \cap d_0(QBrun_{k+2})$ are countably infinitely generated free groups.*

The standard Hall collection process or natural variations can be used to give inductive recipes rather than closed forms for generators. T. Stanford has given a related beautiful exposition of the Hall collection process [17]. The analogous process was applied in [4] to give group theoretic models for iterated loop spaces.

Brunnian braids as well as "almost Brunnian" braids are basic to certain features of homotopy theory. Some details are given in the next section.

7. Homotopy of simplicial groups and Δ-groups

Recall that the pure braid groups P_k form a simplicial group AP_* with $\Gamma_k = P_{k+1}$. Also, recall that the pure braid groups $P_k(S)$ for any connected surface S also yield a Δ-group $\Delta_*(S)$ with $\Delta_k(S) = P_{k+1}(S) = \pi_1(Conf(S, n+1))$ [1].

A simplicial group admits a natural definition of homotopy groups which are analogous to the classical homotopy groups of a topological space. These homotopy groups are defined as follows by Moore [14]. The k-th homotopy group of Γ_* is defined to be the quotient group

$$\pi_k(\Gamma_*) = Z_k/B_k$$

for which

$$Z_k = \cap_{0\leq i\leq k} ker(d_i\colon \Gamma_k \to \Gamma_{k-1}),$$

and

$$B_k = d_o(\cap_{1\leq i\leq k+1} ker(d_i\colon \Gamma_{k+1} \to \Gamma_k)).$$

Brunnian braids as well as "almost Brunnian" braids occur in a fundamental way in the definition of these homotopy groups. The next lemma, a restatement of the definitions, is recorded for convenience.

Lemma 7.1. (1) *There is an isomorphism*

$$\pi_k(\mathrm{AP}_*) \to Brun_{k+1}/d_0(QBrun_{k+2}).$$

Furthermore, $\pi_k(\mathrm{AP}_*)$ is the trivial group.

(2) *There is an isomorphism of left cosets which is natural for pointed embeddings of surfaces S given by*

$$\pi_k(\Delta_*(S)) \to Brun_{k+1}(S)/d_0(QBrun_{k+2}(S)).$$

In the case of a $\Delta-$group, the set of left cosets $\pi_k(\Delta_*) = Z_k/B_k$ may not admit a natural structure of a group which satisfies the property that the natural quotient map $Z_k \to Z_k/B_k$ is a group homomorphism. The case of the Delta-group for the two-sphere

$$S = \mathbb{CP}^1 = S^2$$

is addressed in the next section and is one subject in [1].

8. A PARTIAL SYNTHESIS

Results in [1] give the following.

Theorem 8.1. *If $S = S^2$ and $k \geq 4$, then*

$$\pi_k(\Delta_*(S^2)) = Brun_{k+1}(S^2)/d_0(QBrun_{k+2}(S^2))$$

is a group which is isomorphic to the classical homotopy group $\pi_k(S^2)$.

Furthermore, there is an exact sequence of groups

$$1 \to Brun_{k+2}(S^2) \to Brun_{k+1}(\mathbb{R}^2) \to Brun_{k+1}(S^2) \to \pi_k(S^2) \to 1.$$

The maps $\Theta_n \colon F_k[y_1, \cdots, y_n] \to P_{n+1}$ imply similar consequences. The content of the next theorem is a characterization of the maps Θ_n in terms of the underlying simplicial structure: (1) the maps Θ_n assemble to give a morphism of simplicial groups and (2) the smallest simplicial subgroup of AP_* which contains a generator for P_2 given by $A_{1,2} = \Theta_2(y_1)$ is $F[S^1]$ as proven in [5].

Theorem 8.2. *The exists a unique morphism of simplicial groups*

$$\Theta \colon F[S^1] \to \ \mathrm{AP}_*$$

with $\Theta(y_1) = A_{2,1}$. The map Θ is an embedding. Hence the homotopy groups of $F[S^1]$ are natural sub-quotients of AP_*. Furthermore, the smallest sub-simplicial group of AP_* which contains the element $\Theta(y_1) = A_{1,2}$ is isomorphic to $F[S^1]$.

As a special case of results in [13], it follows that the geometric realization of $F[S^1]$ is homotopy equivalent to the loop space of the 2-sphere ΩS^2. Thus the homotopy groups of $F[S^1]$ regarded as a simplicial group are isomorphic to the homotopy groups of ΩS^2. As a corollary, the connection to the homotopy groups of S^2 are given next.

Corollary 8.3. *The group $\Theta_k(F_k) \cap d_0(QBrun_{k+2})$ is a normal subgroup of $\Theta_k(F_k) \cap Brun_{k+1}$. The quotient group $\Theta_k(F_k) \cap Brun_{k+1}/\Theta_k(F_k) \cap d_0(QBrun_{k+2})$ is isomorphic to $\pi_{k+1}S^2$.*

The E^0-term of a classical spectral sequence for a simplicial group obtained from the descending central series is sometimes known as the Bousfield-Kan spectral sequence. The analogue obtained from the mod-p descending central series is given by the classical unstable Adams spectral sequence for the 2-sphere [6, 18]. The method of proof of

Theorem 3.4 exhibits a close connection between Vassiliev invariants of pure braids and these natural spectrqal sequences.

The next result is recasting Theorem 3.4 proven in [5].

Corollary 8.4. *The maps* $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ *on the level of associated graded Lie algebras*

$$gr_*(\Theta_n)\colon gr_*(F[y_1, y_2, \cdots, y_n]) \to gr_*(P_{n+1})$$

are monomorphisms. Thus the maps Θ_n *induce embeddings on the level of the* E^0*-term of the Bousfield-Kan spectral sequences for* $F[S^1]$ *and* AP_*.

9. Questions

The point of this section is to consider whether the connections between the braid groups and homotopy groups above are useful. Some natural as well as speculative problems are listed next.

1: Find natural methods to distinguish between cosets of certain braids given by homotopy groups rather than braid themselves. For example, Vassiliev invariants of pure braids distinguish all pure braids [11, 12]. Are there weaker versions of Vassiliev invariants which distinguish left cosets in the braid groups given by elements in homotopy groups ?

2: Give combinatorial properties of the natural map $Brun_{k+1}(\mathbb{R}^2) \to Brun_{k+1}(S^2)$ which provides information about the cokernel. A more precise problem is stated next.

Give group theoretic reasons why the order of 2-torsion in $\pi_*(S^2)$ is bounded above by 4 and why the p-torsion for an odd prime p is bounded above by p.

3: The groups $QBrun_{n+2} \cap F_{n+1}$, and $Brun_{n+1} \cap F_n$ are free. Give combinatorial descriptions of the map

$$\Theta_k(F_k) \cap d_0(QBrun_{k+2}) \to \Theta_k(F_k) \cap Brun_{k+1}$$

on the level of abelianizations

$$H_1(\Theta_k(F_k) \cap d_0(QBrun_{k+2})),$$

and

$$H_1(\Theta_k(F_k) \cap Brun_{k+1}).$$

The Serre exact sequence for the homology of a discrete group specializes to

$$H_2(\pi_{k+1}S^2) \to H_1[\Theta_k(F_k) \cap d_0(QBrun_{k+2})]_{\pi_{k+1}S^2} \to H_1[\Theta_k(F_k) \cap Brun_{k+1}] \to \pi_{k+1}S^2 \to \{0\}$$

where A_π denotes the group of coinvariants of a π-module A. Thus $\pi_{k+1}S^2$ is a quotient of the free abelian group $H_1[\Theta_k(F_k) \cap Brun_{k+1}]$ with relations given by the image of $H_1[\Theta_k(F_k) \cap d_0(QBrun_{k+2})]$.

4: Recall that elements in the classical ring of modular forms can be regarded as crossed homomorphisms out of $SL(2, \mathbb{Z})$ with values in the polynomial ring $\mathbb{R}[x_1, x_2]$. The action of $SL(2, \mathbb{Z})$ is specified by the tautological representation on the two dimensional vector space spanned by x_1, x_2 and extended multiplicatively to $\mathbb{R}[x_1, x_2]$ [16].

The natural epimorphism

$$B_3 \to SL(2, \mathbb{Z})$$

composed with these crossed homomorphisms out of $SL(2, \mathbb{Z})$ distinguish some 3-stranded braids. It is natural to ask how these functions distinguish braids.

Determine $H^1(B_{2g+2};\mathbb{R}[x_1,\cdots,x_{2g}])$ where B_{2g+2} acts via a natural symplectic representation on a vector space with basis $\{x_1,\cdots,x_{2g}\}$ the generating module for the polynomial ring and $\mathbb{R}[x_1,\cdots,x_{2g}]$. Describe the sets of braids which satisfy the property that the associated crossed homomorphisms take equal values for these braids.

5: Consider Brunnian braids $Brun_k$. Fix a braid γ with image in Σ_k given by a k-cycle. For any braid α in $Brun_k$, the braid closure of $\alpha\circ\gamma$ is a knot. Describe features of these knots or those obtained from the analogous constructions for $\Theta_k(F_{k-1})\cap Brun_k$. Where do these "fit" in Budney's description of the space of long knots [3] ?

6: Let L denote a Lie algebra which is free as a module over the integers with $Der(L)$ the Lie algebra of derivations. The map $\Theta_k\colon F_k\to P_{k+1}$ induces a map $gr_*(\Theta_k)\colon gr_*(F_k)\to gr_*(P_{k+1})$. Recall the classical adjoint representation

$$ad: L\to Der^{Lie}_*(L)$$

together with the associated map $gr_*(P_{k+1})\to Der(gr_*(F_k))$ obtained by restriction to the Lie algebra generated by $B_{i,k+1}$. Give methods to describe combinatorial properties of the composite

$$gr_*(\Theta_k(F_k)\cap Brun_{k+1})\to gr_*(\Theta_k(F_k)\to gr_*(P_{k+1})\to Der(gr_*(F_k))$$

as well as the image of $gr_*(\Theta_k(F_k)\cap d_0(QBrun_{k+2})$.

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