Cup product and intersection

Michael Hutchings

March 28, 2005

Abstract

This is a handout for my algebraic topology course. The goal is to explain a geometric interpretation of the cup product. Namely, if X is a closed oriented smooth manifold, if A and B are oriented submanifolds of X, and if A and B intersect transversely, then the Poincaré dual of $A \cap B$ is the cup product of the Poincaré duals of A and B. As an application, we prove the Lefschetz fixed point formula on a manifold. As a byproduct of the proof, we explain why the Euler class of a smooth oriented vector bundle is Poincaré dual to the zero section.

1 Statement of the result

A question frequently asked by algebraic topology students is: "What does cup product *mean*?" Theorem 1.1 below gives a partial answer to this question. The theorem says roughly that on a manifold, cup product is Poincaré dual to intersection of submanifolds. In my opinion, this is the most important thing to know about cup product.

To state the theorem precisely, let X be a closed oriented smooth manifold of dimension n. Let A and B be closed oriented smooth submanifolds of X of dimensions $n-i$ and $n-j$ respectively. Assume that A and B intersect transversely. This means that for every $p \in A \cap B$, the map $T_pA \oplus T_pB \to T_pX$ induced by the inclusions is surjective. This condition can be obtained by perturbing A or B. Then $A \cap B$ is a submanifold of dimension $n-(i+j)$, and there is a short exact sequence

$$0 \longrightarrow T_p(A \cap B) \longrightarrow T_pA \oplus T_pB \longrightarrow T_pX \longrightarrow 0.$$

This exact sequence determines an orientation of $A \cap B$. We will adopt the following convention. We can choose an oriented basis $u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j, w_1, \ldots, w_i$ for T_pX such that $u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j$ is an oriented basis for T_pA and $u_1, \ldots, u_{n-i-j}, w_1, \ldots, w_i$ is an oriented basis for T_pB. We then declare that $u_1, \ldots, u_{n-i-j}$ is an oriented basis for $T_p(A \cap B)$. If A and B have complementary dimension, i.e. if $i + j = n$ so that $A \cap B$ is a finite set of points, then a point p is positively oriented if and only if the isomorphism $T_pA \oplus T_pB \simeq T_pX$ is orientation preserving.

Now recall that there is a Poincaré duality isomorphism

$$H^i(M;\mathbb{Z}) \xrightarrow{\simeq} H_{n-i}(M),$$
$$\alpha \longmapsto [M]\cap\alpha.$$

The images of the fundamental classes of A, B, and $A\cap B$ under the inclusions into X define homology classes $[A]\in H_{n-i}(X)$, $[B]\in H_{n-j}(X)$, and $[A\cap B]\in H_{n-i-j}(X)$. We denote their Poincaré duals by $[A]^*\in H^i(X;\mathbb{Z})$, $[B]^*\in H^j(X;\mathbb{Z})$, and $[A\cap B]^*\in H^{i+j}(X;\mathbb{Z})$. We now have:

Theorem 1.1 *Cup product is Poincaré dual to intersection:*

$$[A]^*\smile[B]^*=[A\cap B]^*\in H^{i+j}(X;\mathbb{Z}).$$

This theorem only partially answers the question of what cup product means, because it only works in a smooth manifold, and moreover not every homology class in a manifold can be represented by a submanifold. Nonetheless, this theorem is very useful. Before proving it, we consider some examples.

Example 1.2 Consider the complex projective space $X=\mathbb{C}P^n$. Recall that $\mathbb{C}P^n$ has a cell decomposition with one cell in each of the dimensions $0,2,\ldots,2n$. Thus $H^*(\mathbb{C}P^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ in degrees $0,2,\ldots,2n$, and 0 in all other degrees. Moreover, $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ has a canonical generator α_i, which is the Poincaré dual of a complex $(n-i)$-plane in $\mathbb{C}P^n$, with the complex orientation. Now a generic $(n-i)$-plane intersects a generic $(n-j)$-plane transversely in an $(n-i-j)$-plane with the complex orientation (or the empty set when $i+j>n$). So by Theorem 1.1,

$$\alpha_i\smile\alpha_j=\begin{cases}\alpha_{i+j}, & i+j\le n,\\ 0, & i+j>n.\end{cases}$$

Note that the nondegeneracy of the cup product pairing implies that $\alpha_i\smile\alpha_j=\pm\alpha_{i+j}$, but the above calculation determines the signs (or lack thereof).

Example 1.3 Let $X=T^2=\mathbb{R}^2/\mathbb{Z}^2$. Then $H_1(T^2)\simeq\mathbb{Z}^2$, and we choose generators $[A]$ and $[B]$ where A and B are circles in the x and y directions of $\mathbb{R}^2$, respectively. Also $H_0(T^2)=\mathbb{Z}$ has a canonical generator $[p]$, which is the class of a point p. Let α,β,μ denote the Poincaré duals of $[A],[B],[p]$. Now $A\cap B$ is a positively oriented point, while $B\cap A$ is a negatively oriented point. Thus

$$\alpha\smile\beta=\mu,\qquad \beta\smile\alpha=-\mu.$$

On the other hand,

$$\alpha\smile\alpha=\beta\smile\beta=0.$$

This follows from the sign-commutativity of the cup product. In terms of Theorem 1.1, to compute $\alpha\smile\alpha$ we need to calculate the signed intersection number of two transversely intersecting submanifolds A_1,A_2 representing the class $[A]$.

We cannot take $A_1 = A_2$, but we can take A_1 and A_2 to be parallel, in which case they do not intersect.

Note that the basis $\{[A], [B]\}$ of $H_1(X)$ has a dual basis $\{[A]', [B]'\}$ of $\mathrm{Hom}(H_1(X), \mathbb{Z}) = H^1(X;\mathbb{Z})$ with $\langle [A], [A]' \rangle = \langle [B], [B]' \rangle = 1$ and $\langle [A], [B]' \rangle = \langle [B], [A]' \rangle = 0$. This dual basis does not consist of the Poincaré duals of $[A]$ and $[B]$. Rather, it follows from the above (exercise) that $[A]' = \beta$ and $[B]' = -\alpha$.

In general, if X is a closed oriented manifold we define the *intersection pairing*

$$\cdot : H_{n-i}(X) \otimes H_{n-j}(X) \longrightarrow H_{n-i-j}(X)$$

by applying Poincaré duality, taking the cup product, and then applying Poincaré duality again:

$$\alpha \cdot \beta := [X] \cap (\alpha^* \smile \beta^*).$$

Theorem 1.1 then says that if A and B are transversely intersecting oriented submanifolds representing α and β, then

$$\alpha \cdot \beta = [A \cap B].$$

In particular, if $\dim(A) + \dim(B) = \dim(X)$ and X is path connected, then $\alpha \cdot \beta \in H_0(X) = \mathbb{Z}$ is the signed number of intersection points $\#(A \cap B)$. When X is not connected, we usually interpret $\alpha \cdot \beta$ to be the image of this element of $H_0(X)$ under the augmentation map $H_0(X) \to \mathbb{Z}$, i.e. the total signed number of fixed points.

There is also an obvious analogue of Theorem 1.1 for unoriented manifolds using $\mathbb{Z}/2$ coefficients.

2 The Lefschetz fixed point theorem

A more interesting application of Theorem 1.1 is given by the following version of the Lefschetz fixed point theorem.

Let X be a closed smooth manifold and let $f : X \to X$ be a smooth map. A *fixed point* of f is a point $p \in X$ such that $f(p) = p$. The fixed point p is *nondegenerate* if

$$1 - df_p : T_pX \to T_pX$$

is invertible. If p is nondegenerate, we define the *Lefschetz sign* $\epsilon(p) \in \{\pm 1\}$ to be the sign of $\det(1 - df_p)$. It is a fact, which we will not prove here, that if f is "generic", then all the fixed points are nondegenerate, in which case (by Lemma 2.3 below) there are only finitely many of them. In this situation we define the signed count of fixed points

$$\#\,\mathrm{Fix}(f) := \sum_{f(p)=p} \epsilon(p) \in \mathbb{Z}.$$

The Lefschetz theorem then says:

Theorem 2.1 *Let f be a closed smooth manifold and let $f : X \to X$ be a smooth map with all fixed points nondegenerate. Then*

$$\#\operatorname{Fix}(f) = \sum_i (-1)^i \operatorname{Tr}(f_* : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q}))\,.$$

Note that it follows from the universal coefficient theorem that the above traces are integers.

Example 2.2 If f as above is homotopic to the identity, then $\#\operatorname{Fix}(f) = \chi(X)$. In particular, a vector field V on X gives rise to a perturbation f of the identity, whose fixed points are the zeroes of V. So if the zeroes of V are appropriately nondegenerate, then they can be counted with suitable signs so that

$$\#V^{-1}(0) = \chi(X).$$

To prove the Lefschetz theorem, we will do intersection theory in $X \times X$. We assume for now that X is orientable, although this assumption can be removed, see below. Define the *diagonal*

$$\Delta := \{(x,x) \mid x \in X\} \subset X \times X.$$

Also, define the *graph*

$$\Gamma(f) := \{(x, f(x)) \mid x \in X\} \subset X \times X.$$

There is an obvious bijection between fixed points of f and intersections of the graph and the diagonal. More precisely, we have:

Lemma 2.3 *f has nondegenerate fixed points if and only if the graph and the diagonal intersect transversely. In that case,*

$$\#\operatorname{Fix}(f) = [\Gamma(f)] \cdot [\Delta].$$

Proof. Exercise. □

Assuming nondegenerate fixed points, we now compute the intersection number $[\Gamma(f)] \cdot [\Delta]$. This requires the following lemma about intersection theory in $X \times X$.

Lemma 2.4 *Let $\alpha, \beta, \gamma, \delta \in H_*(X)$. Then:*

(a) $(\alpha \times \beta) \cdot (\gamma \times \delta) = (-1)^{|\beta||\gamma|}(\alpha \cdot \gamma) \times (\beta \cdot \delta)$.

(b) $[\Gamma(f)] \cdot (\alpha \times \beta) = (-1)^{|\alpha|} f_*\alpha \cdot \beta$.

Here $|\alpha|$ denotes the dimension of α. The lemma is pretty obvious (up to checking signs) if $\alpha, \beta, \delta, \gamma$ can be represented by submanifolds. In general, the lemma follows from basic properties of cap and cross products, and we leave the details as an exercise.

Now let $\{e_k\}$ be a basis for the vector space $H_*(X;\mathbb{Q})$, consisting of elements of pure degree. Let $\{e'_k\}$ be the dual basis of $H_*(X;\mathbb{Q})$, with respect to the intersection pairing. That is, if $e_k \in H_m(X;\mathbb{Q})$, then $e'_k \in H_{n-m}(X;\mathbb{Q})$ satisfies $e_j \cdot e'_k = \delta_{i,j}$.

Lemma 2.5 $[\Delta] = \sum_k e_k \times e'_k$.

Proof. By the Künneth theorem, $H_*(X \times X; \mathbb{Q}) = H_*(X; \mathbb{Q}) \otimes H_*(X; \mathbb{Q})$, and since the intersection pairing is perfect, it is enough to check that both sides of the equation have the same intersection pairing with $e'_i \times e_j$ when $|e'_i| + |e_j| = n$. By Lemma 2.4 with $f = \mathrm{id}_X$, we have

$$\begin{aligned}\left(\sum_k e_k \times e'_k\right) \cdot (e'_i \times e_j) &= \sum_k (-1)^{|e'_k||e'_i|} (e_k \cdot e'_i) \times (e'_k \cdot e_j) \\ &= \sum_k (-1)^{|e'_i|} e'_i \cdot e_j \\ &= [\Delta] \cdot (e'_i \times e_j).\end{aligned}$$

□

We can now complete the proof of the Lefschetz theorem:

$$\begin{aligned}\#\,\mathrm{Fix}(f) &= [\Gamma(f)] \cdot [\Delta] \\ &= [\Gamma(f)] \cdot \sum_k e_k \times e'_k \\ &= \sum_k (-1)^{|e_k|} f_* e_k \cdot e'_k \\ &= \sum_i (-1)^i \,\mathrm{Tr}\left(f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})\right).\end{aligned}$$

Exercise 2.6 What can we do if X is not orientable? (Hint: see handout on homology with local coefficients.)

3 The Thom isomorphism theorem

The proof of Theorem 1.1 uses the Thom isomorphism theorem, which we now state. Henceforth we assume some familiarity with vector bundles.

Let E be a real vector bundle over B with a metric. Define D to be the associated disk bundle over B, consisting of vectors of length ≤ 1. Define S to be the associated sphere bundle over B, consisting of vectors of length equal to 1.

Definition 3.1 Let $\pi : E \to B$ be an oriented rank n real vector bundle. A *Thom class* for E is a cohomology class $u \in H^n(D, S; \mathbb{Z})$ such that for every $x \in B$, the restriction of u to $H^n(D_x, S_x; \mathbb{Z}) \simeq \mathbb{Z}$ is the prefered generator determined by the orientation.

The following is one version of the Thom isomorphism theorem.

Theorem 3.2 *Let $\pi : E \to B$ be an oriented rank n real vector bundle with a metric. Then:*

(a) E has a unique Thom class $u \in H^n(D,S;\mathbb{Z})$.

(b) The map

$$\begin{aligned} H^k(B;\mathbb{Z}) &\longrightarrow H^{k+n}(D,S;\mathbb{Z}), \\ \alpha &\longmapsto \pi^*\alpha \smile u \end{aligned}$$

is an isomorphism.

Proof. The Thom isomorphism theorem can be proved using a Mayer-Vietoris induction argument (for the case when B is compact), followed by a tricky direct limit argument (for general B). For details see Milnor and Stasheff, chapter 9. Alternatively, see the handout on spectral sequences. □

Remark 3.3 The Thom class can also be regarded as an element of $H^n(E, E\setminus B;\mathbb{Z})$, and from this point of view the Thom isomorphism theorem works without a metric. One can replace the coefficient ring $\mathbb{Z}$ by any ring R with unit. If $R = \mathbb{Z}/2$, an orientation is not needed.

Intuitively, the Thom class u, evaluated on an n-chain α, counts the intersections of α with the zero section. Lemma 4.1 below is a precise version of this intuition, as will eventually become clear.

4 Proof of the main theorem

We now prove Theorem 1.1. The proof here is based on material in the books by Bredon and Milnor-Stasheff, with some modifications.

Recall that if M is a closed oriented n-manifold with boundary, then there is a relative fundamental class $[M] \in H_n(M,\partial M)$, and a Poincaré-Lefschetz duality isomorphism

$$\begin{aligned} H^i(M,\partial M;\mathbb{Z}) &\xrightarrow{\simeq} H_{n-i}(M;\mathbb{Z}), \\ \alpha &\longmapsto [M]\cap\alpha. \end{aligned}$$

The following lemma says that in the smooth case, the Thom class is just the Poincaré-Lefschetz dual of the zero section. If A is a subspace of X, let $\imath_A^X : A \to X$ denote the inclusion.

Lemma 4.1 *Let B be a closed smooth oriented k-manifold, and let E be a smooth rank n oriented real vector bundle over B. Then*

$$\left(\imath_B^D\right)_*[B] = [D]\cap u \in H_k(D).$$

Here E is given any metric; we know that a metric exists. Also B is regarded as a submanifold of D via the zero section. Finally, the orientation on D is determined by the orientations of the fibers and the base, in that order. (It would perhaps be more usual to use the other order, but then there would be more minus signs in our formulas.)

Proof. Without loss of generality, B is connected. We now have isomorphisms

$$\mathbb{Z} = H^0(B;\mathbb{Z}) \stackrel{\pi^*(\cdot)\smile u}{\longrightarrow} H^n(D,\partial D;\mathbb{Z}) \stackrel{[D]\cap}{\longrightarrow} H_k(D) \stackrel{\pi_*}{\longrightarrow} H_k(B) = \mathbb{Z}.$$

The generator $1 \in H^0(B;\mathbb{Z})$ maps to $[D] \cap u \in H_k(D)$, and this must map to a generator of $H_k(B)$. Since π_* is an isomorphism on homology, it follows that $[D] \cap u = \pm \left(\imath_B^D\right)_* [B]$.

Unfortunately, I do not know a very satisfactory way to nail down the sign from this point of view, since the details of the definition of the cap/cup product have been abstracted away. One approach is to pass to real coefficients and use de Rham cohomology, where the cup product is just wedge product of differential forms, and so the sign is easily understood in terms of the orientations. See for instance the book by Bott and Tu. □

Returning to the setting of the main theorem, let $N_A^X = N$ be a tubular neighborhood of A, which we can regard as an oriented rank i vector bundle over A. The Thom class of A can be regarded as an element

$$u_A \in H^i(N, N \setminus A;\mathbb{Z}).$$

Let u_A^X denote the image of u_A under the composition

$$H^i(N, N \setminus A;\mathbb{Z}) \longrightarrow H^i(X, X \setminus A; Z) \longrightarrow H^i(X;\mathbb{Z}).$$

Here the first map is the inverse of the excision isomorphism.

Lemma 4.2 $[A]^* = u_A^X \in H^i(X;\mathbb{Z})$.

Proof. Since Poincaré duality is an isomorphism, we can equivalently prove

$$\left(\imath_A^X\right)_* [A] = [X] \cap u_A^X \in H_{n-i}(X). \qquad (1)$$

To do so, consider the commutative diagram

$$\begin{array}{ccccc}
H_n(X) \otimes H^i(X, X \setminus A) & \longrightarrow & H_n(X, X \setminus A) \otimes H^i(X, X \setminus A) & \longrightarrow & H_n(N, N \setminus A) \otimes H^i(N, N \setminus A) \\
\downarrow & & \downarrow \cap & & \downarrow \cap \\
H_n(X) \otimes H^i(X) & \stackrel{\cap}{\longrightarrow} & H_{n-i}(X) & \longleftarrow & H_{n-i}(N)
\end{array}$$

where all cohomology is with integer coefficients, and the upper right arrow is the tensor product with an inverse excision isomorphism and an excision isomorphism. By Lemma 4.1, we know that $[N] \otimes u_A$ in the upper right maps down to $\left(\imath_A^X\right)_* [A] \in H_{n-i}(X)$. But $[X] \cap u_A^X$ in the lower left maps to the same place. This follows from commutativity of the diagram, together with the fact that $[X]$ maps to $[N]$ under the composition

$$H_n(X) \longrightarrow H_n(X, X \setminus A) \stackrel{\simeq}{\longleftarrow} H_n(N, N \setminus A).$$

This last fact follows from the defining property of the fundamental class. $\square$

We can now prove Theorem 1.1. To start, our orientation conventions imply that there is an isomorphism of oriented vector bundles

$$N^A_{A\cap B} = \left(N^X_B\right)|_{A\cap B}.$$

It follows from the characterizing property of the Thom class that

$$u^A_{A\cap B} = \left(\imath^X_A\right)^* u^X_B. \tag{2}$$

Now to prove Theorem 1.1, by Lemma 4.2 it is enough to show that $u^X_{A\cap B} = u^X_A \smile u^X_B$, or equivalently (since Poincaré duality is an isomorphism),

$$[X] \cap u^X_{A\cap B} = [X] \cap (u^X_A \smile u^X_B).$$

Using equations(1) and (2), we compute

$$\begin{aligned}
[X] \cap u^X_{A\cap B} &= \left(\imath^X_{A\cap B}\right)_* [A\cap B] \\
&= \left(\imath^X_A\right)_* \left(\imath^A_{A\cap B}\right)_* [A\cap B] \\
&= \left(\imath^X_A\right)_* \left([A] \cap u^A_{A\cap B}\right) \\
&= \left(\imath^X_A\right)_* \left([A] \cap \left(\imath^X_A\right)^* u^X_B\right) \\
&= \left(\imath^X_A\right)_* [A] \cap u^X_B \\
&= \left([X] \cap u^X_A\right) \cap u^X_B \\
&= [X] \cap \left(u^X_A \smile u^X_B\right).
\end{aligned}$$

5 The Euler class and the zero section

If $E \to B$ is an oriented rank n real vector bundle, one can define the *Euler class*

$$e(E) \in H^n(B;\mathbb{Z})$$

to be the image of the Thom class u under the composition

$$H^n(E, E\setminus B;\mathbb{Z}) \to H^n(E;\mathbb{Z}) \simeq H^n(B;\mathbb{Z}).$$

It follows easily from the definition that e is natural, and that $e(E) = 0$ if E has a nonvanishing section. If B is a CW complex, then it can be shown that $e(E)$ is the primary obstruction to the existence of a nonvanishing section. A related fact is that the Euler class of a smooth vector bundle is Poincaré dual to the zero set of a generic section, and we can now prove this using the machinery introduced above.

Let ψ be a section of E, let Γ denote the graph of ψ, and let $Z = \psi^{-1}(0) = \Gamma \cap B$ denote the zero set of ψ. Suppose that the bundle E is smooth. Then it can be shown that for a generic section ψ, the graph Γ is transverse to the zero section B, in which case it follows that the zero set Z is a submanifold of

B. Also, the derivative of ψ along the zero section defines an isomorphism of vector bundles

$$N_Z B \simeq E|_Z, \tag{3}$$

and we use this to orient Z.

Theorem 5.1 *Let $E \to B$ be a smooth, oriented rank n real vector bundle over a smooth closed oriented manifold B, and let ψ be a generic section with zero set Z. Then*

$$e(E) = [Z]^* \in H^n(B; \mathbb{Z}).$$

Proof. We have

$$[Z]^* = u_Z^B = \left(\imath_B^D\right)^* \left(u_B^D\right) = e(E).$$

Here the first equality is Lemma 4.2, the second equality follows from the isomorphism (3), and the third equality follows from the definition of the Euler class. □

Example 5.2 If M is a closed oriented smooth n-manifold, then it follows from Theorem 5.1 and Example 2.2 that

$$e(TM) = \chi(M)[pt]^* \in H^n(M; \mathbb{Z}).$$