Contemporary Mathematics
Volume **00**, 19xx

# From operads to 'physically' inspired theories

Jim Stasheff

An operad-chik looks at configuration spaces, moduli spaces and mathematical physics

## 1. Introduction

As evidenced by these conferences (Hartford and Luminy), operads have had a renaissance in recent years for a variety of reasons. Originally studied entirely as a tool in homotopy theory, operads have recently received new inspirations from homological algebra, category theory, algebraic geometry and mathematical physics. I'll try to provide a transition from the foundations to the frontier with mathematical physics.

For me, the transition occurred in two stages. First, there is the generalization of Lie algebra cohomology known as BRST (Becchi-Rouet-Stora-Tyutin) cohomology, which turned out to be very closely related to strong homotopy Lie ($L_\infty$) algebras, which I will describe later in homological algebraic terms - along the lines of Balavoine's talk at this conference [**5**]. That description makes no use of operads, but the relevance of operads appeared later in the work of Hinich and Schechtman [**25**].

Operads revealed themselves as more directly involved in String Field Theories (SFTs) and Vertex Operator Algebras (VOAs), both of which in turn draw on Conformal Field Theories (CFTs). Central to CFTs are algebraic structures parameterized by moduli spaces $\mathcal{M}_{g,n}$ of $n$-punctured Riemann surfaces of genus $g$. Operads are almost visible

1991 *Mathematics Subject Classification.* Primary 55P99, 18G99, 81R99, 81T70, 81T70 Secondary 57T99, 18G55, 81R50.

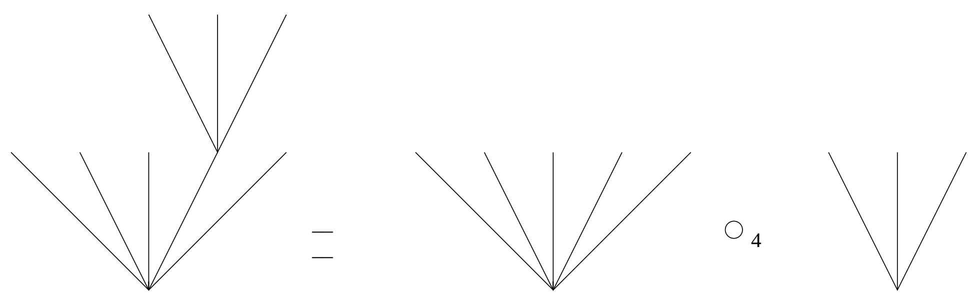
Research supported in part by the NSF throughout most of my career, currently under grant DMS-9504871.

in $\mathcal{M}_{0,n+1}$ if we choose one point "$\infty$" as outgoing and the other $n$ as incoming.

## 2. Background

But perhaps I should back up and recall the essentials of operads. At the front of this volume appears the formal, official definition due to Peter May [**42**] as well as the alternate, assuming there is an 'identity' in $\mathcal{C}(1)$, in terms of $\circ_i$ operations, cf. Gerstenhaber's comp algebras [**19**]. (For a full discussion of the appropriate definitions and relations in the absence of an 'identity' in $\mathcal{C}(1)$, see [**40**].) In my talk at Luminy on the pre-history of operads [**48**], I emphasized planar rooted trees. For operads proper, rooted trees form the 'mother of all operads'.

**2.1. The tree operad.** Let $\mathcal{T}(n)$ be the set of trees with 1 root and $n$ leaves labelled 1 through $n$. The collection $\{\mathcal{T}(n)\}$ of tree spaces forms a non-symmetric operad (or comp algebra): given trees $f \in \mathcal{T}(k), g \in \mathcal{T}(j)$, for each $i$, let $f \circ_i g$ be the tree obtained by grafting the root of $g$ to the leaf of $f$ labelled $i$. More generally, given trees $f \in \mathcal{T}(k), g_i \in \mathcal{T}(n_i)$, let $\gamma(f; g_1, \dots, g_k) \in \mathcal{T}(\Sigma n_i)$ be the tree obtained by grafting the root of $g_i$ to the leaf of $f$ labelled i.




This and the basic examples that appear in the introduction to this volume [**43**]and in [**48**] are discrete or topological operads. To move to (vector space) algebra from discrete operads, just take the vector spaces spanned by the discrete sets. To move to differential graded algebra from topological operads, suitable chain functors serve well. For any topological operad $\mathcal{C}$, the singular chains $C_*(\mathcal{C}(n))$ form a dg (differential graded) operad. Since the associahedra are presented as cell complexes and the $\circ_i$ operations are cellular, the cellular chains $\mathcal{A}(n) = CC_*(K_n)$ form a dg operad $\mathcal{A}$.

**2.2. $A_\infty$-algebras.** An algebra over $\mathcal{A}$ is known as an $A_\infty$-algebra (or strongly homotopy associative algebra) and consists of a graded module $V$ with maps

$$m_n : V^{\otimes n} \to V \text{ of degree } n-2$$

satisfying suitable compatibility conditions. In particular,

$m_1 = d$ is a differential,

$m = m_2 : V \otimes V \to V$ is a chain map, that is, $d$ is a derivation with respect to $m = m_2$,

$m_3 : V^{\otimes 3} \to V$ is a chain homotopy for associativity, i.e.

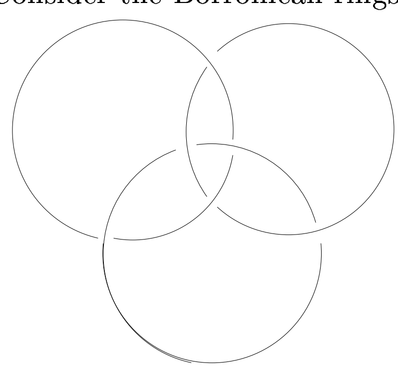
$$dm_3 + m_3 d = m(m \otimes 1) - m(1 \otimes m),$$

$m_4$ is a 'higher homotopy' such that $dm_4 - m_4 d$ equals a sum of five terms, the 'pentagonal' relation which would be precisely zero in Drinfel'd's quasi-Hopf algebras [**17**]. (The homotopy interpretation exists, for example, in the classifying space of the category of representations of a quasi-Hopf algebra [**27**].)

The original and motivating example of an $A_\infty$-algebra was provided by the singular chains on a based loop space $\Omega X$.

**Warning!** When $d = 0$ (for example, after passing to (co)homology from (co)chains), the conditions give a strictly associative algebra **but** there may still be non-trivial maps $m_n$. Here are two natural examples:

EXAMPLE 2.1. Consider the Borromean rings



Borromean rings

consisting of three circles which are pairwise unlinked but all together are linked. (The name 'Borromean' derives from their appearance in the coat of arms of the House of Borromeo in Italy.) If we regard them as situated in $S^3$, then the cohomology ring of the complement is a trivial algebra, but $m_3$ is non-zero in cohomology, being represented by Massey products and detecting the simultaneous linking of all three circles [**41**].

The relevant Massey product is defined as follows: Let $u, v, w$ be cocyles which are Alexander dual to the fundamental homology classes of the three circles. Because the circles are pairwise unlinked, the corresponding cocycles in the complement have cup products which are cohomologically trivial, e.g. $u \cup v = \delta a$ and $v \cup w = \delta b$. Massey shows that the triple product $< u, v, w >$ represented by $ub + aw$ generates $H^1$ of the complement.

EXAMPLE 2.2. Consider the homogeneous space $\mathrm{Sp}(5)/\mathrm{SU}(5)$ where $\mathrm{Sp}(5)$ denotes the 'orthogonal' group in 5 quaternionic dimensional space. According to Borel's calculations [**13**], the cohomology algebra has a single generator in each of dimensions 6, 10, 21 and 25 with only the pairing to the fundamental class in dimension 31 being non-trivial. This is isomorphic to the cohomology algebra of the connected sum of $S^6 \times S^{25}$ and $S^{10} \times S^{21}$, but these spaces have distinct homotopy types, even rationally, since triple Massey products are trivial in the connected sum but non-trivial in the homogeneous space. Borel calculates the cohomology via the Leray-Serre spectral sequence of the fibration

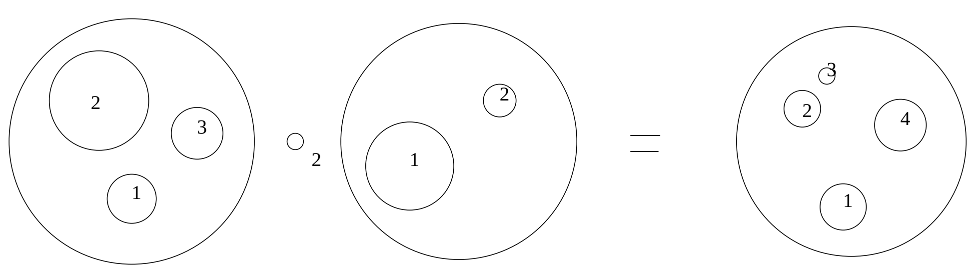
$$\mathrm{Sp}(5) \to \mathrm{Sp}(5)/\mathrm{SU}(5) \to \mathrm{BSU}(5)$$

and the Massey product calculation is manifest, though not named as such.

## 3. Configuration spaces

Now let's return briefly to the moduli space $\mathcal{M}_{0,n+1}$. 'Genus 0' is called in physspeak 'tree level', for reasons which should become apparent. The modular equivalence relation allows us to choose representatives in which the first two points are $0, 1 \in \boldsymbol{C}P(1)$ and the last is '$\infty$', thus identifying $\mathcal{M}_{0,n+1}$ with the configuration space $F(R^2 - \{0,1\}, n-2)$. Configuration spaces do not form an operad, but there are two ways to get an operad from the configuration spaces:

**3.1. The little disks operad.** First, decorate the chosen points with non-overlapping 'little' disks. Let $D$ be the unit disk in the plane and let $\mathcal{D}(n)$ be the space of all maps $f$ from $\coprod_{i=1}^n D \to D$ such that $f$, when restricted to each disk, is the composition of translation and multiplication by a positive real number and images of the components of $f$ are disjoint.



Of course, this generalizes to little $n$-balls for any $n$. For certain purposes, little $n$-cubes [**12**] are more useful [**42**].

**3.2. Compactifications.** Instead of adding decorations to the configurations, we can obtain an operad by suitable compactification of configuration space. This is the approach relevant to many of the applications to physics and to knot invariants.

In my Hartford talk, I tried to span 162 years of mathematics, from Gauss' linking number to some recent developments involving the concept of operad. That sketch of the historical development appears as Appendix A to this paper. Here I will go directly to the 'cyclohedra', the cousins of the associahedra which are relevant to knot invariants. (The portion of my Hartford talk which summarized the paper of Bott and Taubes [**14**] is omitted in deference to the original. Some updates due to Labastida and to Dylan Thurston are included in Appendix A here.)

To generalize Gauss' approach for links to knots, we can try the configuration space of pairs of distinct points on a single circle, but then the space is not compact. One way around this difficulty, due to Calugareanu in 1959 [**15**] and in an improved version to Pohl in 1968 [**45**], uses a 'nearby' parallel $K'$, that is a framing of $K$. The new approach due to Kontsevich [**34**] involves the compactification of the configuration space of $n$ distinct points on a circle as a manifold-with-corners, keeping track of how points on $S^1$ approach coincidence.

The way to retain that information is well known in algebraic geometry under the rubric of 'blowing up' and has been worked out in great detail for configurations on non-singular algebraic varieties by Fulton and MacPherson [**18**]. The real, as opposed to complex, analog has been studied by Axelrod and Singer [**4**]; not only do we want to work over the real instead of the complex field but we also do not want to projectivize the information. The essential idea is to compactify by adding a boundary where two (or more) points collide. For example, in $\mathbb{R}^3$, where two points collide, adjoin the unit sphere bundle of this subspace of $(\mathbb{R}^3)^n$.

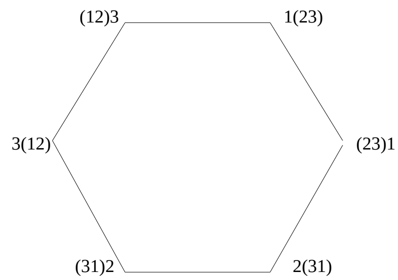
Bott and Taubes looked at the details in the case of a circle, saw the result as the product of a circle $S^1$ with a polytope and drew pictures of the polytopes for 2, 3 and 4 points, although for more points they were content with the description in terms of blow-ups. In his lecture at MSRI, Bott drew the pictures and mentioned that the polytopes resembled the ones I had constructed for studying loop spaces some 34 years before. On closer examination I realized that my polytopes, since dubbed the associahedra, appeared as faces of the Bott-Taubes polytopes $W_n$, which I will call cyclohedra. (I have reindexed the polytopes by the number of points in the relevant configurations, rather than by their dimensions.) Further analysis provided a simple description of the compactifications as truncations of the simplex. This led me to reconsider the associahedra and describe their realization also as truncated simplices - something I had not noticed for 35 years! In discussing these matters with Shnider and Sternberg, I found that not only do they describe the associahedra as truncated simplices in their book, but the truncation appears (somewhat obscurely - at least to me) on the cover! (Further details of this truncation are in Appendix B, which is joint work with Shnider.)

## 4. Cyclohedra

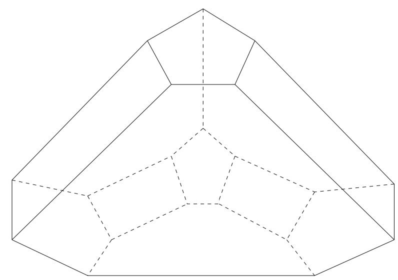
The polytopes $W_n$ which realize the compactification of $F(S^1, n)$ as $S^1 \times W_n$ I suggest calling **cyclohedra.** (Bott and Taubes use the notation $C(n, S^1)$ for $F(S^1, n)$ - see Appendix A.) Parameterize the circle as usual by arclength normalized to have circumference 1 (or $2\pi$ if you insist). Number the points $x_0, \ldots, x_{n-1}$ in order of increasing parameter. Let $\theta$ be the parameter for $x_0$ and let $t_i$ be the arclength from $x_i$ to $x_{i+1}$ with $t_{n-1}$ the arclength from $x_{n-1}$ to $x_0$. Noticing that $\Sigma t_i = 1$, the space $F(S^1, n)$ is seen to be homeomorphic to $S^1 \times \mathring{\Delta}^{n-1}$, the product of the circle with the open $(n-1)$-simplex. The cyclohedra $W_n$ are the 'blow-up' compactifications of $\mathring{\Delta}^{n-1}$ very much as the associahedra $K_n$ are for the interval, except that $x_n$ is interpreted as $x_0$. Thus they can be viewed also as truncations of simplices - in fact the only change is that since $x_0$ can run into $x_{n-1}$, the truncations include additionally the same formulas but with the indices interpreted *mod n*. (See Appendix B where the indices are subintervals $I = (i, \ldots, j)$ of $(1, \ldots, n-1)$, to be interpreted cyclically for the cyclohedron, e.g. $(n-1, 1, 2)$.)

Just as the associahedron $K_n$ can be described as a convex polytope with one vertex for each way of inserting parentheses in a meaningful way in a word of $n$ letters, so the cyclohedron $W_n$ has vertices which are given by all meaningful ways of inserting parentheses in a string of $n$ symbols arranged on a circle.


$W_2$




$W_3$



$W_4$

The facets of $W_n$ are of the following forms:

$$W_{n-1}$$

$$K_n$$

$$W_r \times K_s \text{ with } r+s-1=n$$

Faces of lower dimension are of the form

$$W_k \times K_{n_1} \times \cdots \times K_{n_k} \text{ with } \Sigma\, n_i = n.$$

That is, we have various inclusions:

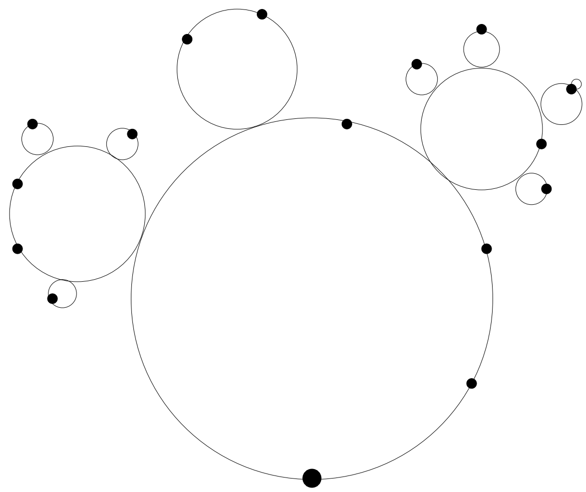
$$W_k \times K_{n_1} \times \cdots \times K_{n_k} \hookrightarrow \partial W_n,$$

just as we had

$$K_k \times K_{n_1} \times \cdots \times K_{n_k} \hookrightarrow \partial K_n$$

for the associahera.

In my work with Kimura and Voronov on string and conformal field theories [**31, 32**], we were concerned with the (Deligne-)Knudsen(-Mumford) compactification of moduli spaces of punctured Riemann surfaces and their way of visualing points on the compactification divisor in terms of '$n$-punctured stable curves'. We found an analogous representation of points on the boundary of a real compactification of moduli spaces as stable Riemann surfaces with pairs of tangent directions at double points mod diagonal rotations [**31**]. Then Kimura and I found analogous pictures for the associahedra and the cyclohedra. A schematic picture of a point on the boundary of $\overline{F}(S^1, n)$ is given by a 'tree' of circles attached at double points with punctures distinct from the double points.



Observe that (the compactification of) the moduli space of $n$ distinct points on the real line, modulo translation and dilation which can fix the first and last points as 0 and 1, is diffeomorphic to (the compactification of) the configuration space $F(\mathring{I}, n-2)$. The biggest circle is to be interpreted as an element of $F(S^1, k)$ while the smaller circles are to be interpreted as elements of $F(\mathring{I}, n_i - 2)$ with the end points 0 and 1 identified with a puncture on the adjacent circle. For the special case in which only one $n_i$ is different from 2, we have a representation of a facet of the form

$$W_r \times K_s \quad \text{with} \quad r + s - 1 = n.$$

The inclusions

$$K_k \times K_{n_1} \times \cdots \times K_{n_k} \hookrightarrow \partial K_n$$

form the structure maps of an operad, while the cyclohedra have facets of the form

$$W_k \times K_{n_1} \times \cdots \times K_{n_k} \hookrightarrow \partial W_n$$

and so do not form an operad, but rather what Markl [**40**] has recently described and dubbed a *module over an operad* (which is not the same as a module over an algebra over an operad). That is, we can regard the facet inclusions for both $K_n$ and $W_n$ as $\circ_i$ operations and then compose them appropriately. We have compatibility in the sense of commutativity of the following diagram:

$$\begin{array}{ccc} W_p \times K_q \times K_r & \longrightarrow & W_p \times K_{q+r-1} \\ \downarrow & & \downarrow \\ W_{p+q-1} \times K_r & \longrightarrow & W_{p+q+r-2} \end{array}$$

where the arrows are given by suitable $\circ_i$ operations.

Although the cyclohedra first arose in the context of compactified configuration spaces, their structure as a module over the operad of associahedra suggests other questions, perhaps the most intriguing of which are the subjects of the next section.

**4.1. Approximations.** The operads originally invented by Peter May for studying iterated loop spaces were used in a variety of ways. For the associahedra, we have the following construction:

Just as we have degeneracy maps $s_i : \Delta_n \to \Delta_{n-1}$ for $i = 0, 1, \ldots, n-1$, so I defined (with more effort) degeneracy maps $s_i : K_n \to K_{n-1}$ [**49**]. For a space $X$ with distinguished point $*$, form the space

$$KX := \coprod K_n \times X^n / \sim$$

where $(s_i(t), x_1, \ldots, x_{n-1}) \sim (t, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_{n-1})$ for $t \in K_n$.

[Corollary to a Theorem of May [**42**]]: For a connected CW complex $X$, the space $KX$ has the homotopy type of $\Omega\Sigma X$, the based loop space on the suspension of $X$.

What about $WX$, similarly defined using the cyclohedra instead of the associahedra? That is, we can define (again with effort) degeneracy maps $s_i : W_n \to W_{n-1}$ and form

$$WX := \coprod W_n \times X^n / \sim$$

where $(s_i(t), x_1, \ldots, x_{n-1}) \sim (t, , x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_{n-1})$ for $t \in W_n$.

The module structure of the $W_n$ over the $K_n$ as operad means that we can define maps $WKX \to WX$ and further that we have commutativity of the diagram

$$\begin{array}{ccc} WKKX & \longrightarrow & WKX \\ \downarrow & & \downarrow \\ WKX & \longrightarrow & WX. \end{array}$$

The space $KX$ is an approximation to $\Omega\Sigma X$, but what does $WX$ approximate?

*Added in 1996:* Markl has observed that there are also 'degeneracies' $W_n \to K_n$ which are deformation retractions compatible with the facet structure, thus $WX$ is also an approximation to $\Omega\Sigma X$. This makes the following the more interesting question.

Since the cyclic symmetry of $F(S^1, n)$ extends to an action of the cyclic group $\boldsymbol{Z}/n$ on $W_n$ and we can define the degeneracy $s_0$ first and then use the action of $\boldsymbol{Z}/n$ to define the other $s_i$'s, we could also construct

$$W_{\boldsymbol{Z}}X := \coprod W_n \times_{\boldsymbol{Z}/n} X^n / \sim$$

or yet again

$$CWX := \coprod \overline{F}(S^1, n) \times_{\boldsymbol{Z}/n} X^n / \sim .$$

It is unknown what the analog of May's theorem should be for these constructions; we have approximations to as yet unknown functors.

## 5. $L_\infty$-algebras and BRST cohomology

Although $A_\infty$-spaces arose first and led to $A_\infty$-algebras via their chain algebras, the Lie analogs occurred first as algebras [**46**] and the relevant topology much later.

### 5.1. $L_\infty$-algebras or (Strong) Homotopy Lie algebras.

Definition 5.1 ($L_\infty$ or (Strong) Homotopy Lie algebras). An *$L_\infty$-algebra* is a graded vector space $V = \sum_{i\in\mathbb{Z}} V_i$ with a differential $Q$ of degree 1 (with $Q^2 = 0$) and a collection of $n$-ary brackets:

$$[v_1, \ldots, v_n] \in V, \qquad v_1, \ldots, v_n \in V,\ n \geq 2,$$

which are homogeneous of degree $3 - 2n$ and super (or graded) symmetric:

$$[v_1, \ldots, v_i, v_{i+1}, \ldots, v_n] = (-1)^{|v_i||v_{i+1}|}[v_1, \ldots, v_{i+1}, v_i, \ldots, v_n],$$

$|v|$ denoting the degree of $v \in V$, and satisfy the relations

$$\text{(5.5.1)} \quad Q[v_1, \ldots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \ldots, Qv_i, \ldots, v_n] = \sum_{k+l=n+1} \sum_{\text{unshuffles } \sigma} \epsilon(\sigma)[[v_{i_1}, \ldots, v_{i_k}], v_{j_1}, \ldots, v_{j_{l-1}}],$$

where $\epsilon(i) = (-1)^{|v_1|+\cdots+|v_{i-1}|}$ is the sign picked up by taking $Q$ through $v_1, \ldots, v_{i-1}$ and, for the unshuffle $\sigma : \{1, 2, \ldots, n\} \mapsto \{i_1, \ldots, i_k, j_1, \ldots, j_{l-1}\}$, the sign $\epsilon(\sigma)$ is the sign picked up by the elements $v_i$ passing through the $v_j$'s during the unshuffle of $v_1, \ldots, v_n$, as usual in superalgebra.

REMARK 5.1. Here we follow the physics grading and sign conventions in our definition of a homotopy Lie algebra [**52, 53**]. These are equivalent to but different from those in the existing mathematics literature, cf. Lada and Stasheff [**36**], in which the $n$-ary bracket has degree $2 - n$. With those mathematical conventions, homotopy Lie algebras occur naturally as deformations of Lie algebras. If $L$ is a Lie algebra and $V$ is a complex with a homotopy equivalence to the trivial complex $0 \to L \to 0$, then $V$ is naturally a homotopy Lie algebra; see Schlessinger and Stasheff [**46**], also Getzler and Jones [**21**]. Similarly, with the physics conventions, $L_\infty$-algebras can occur naturally as deformations of "graded Lie algebras with a bracket of degree $-1$", which are equivalent to ordinary graded Lie algebras after a shift of grading and redefining the bracket by a sign, see [**53**, Section 4.1]. (For topologists, the physics conventions correspond to the algebra of homotopy groups of a space with respect to Whitehead product while the math conventions correspond to the algebra of homotopy groups of a loop space with respect to Samelson product. Indeed, $L_\infty$-algebras first appeared implicitly in Sullivan's minimal models of rational homotopy types, as evidenced in the following alternative.)

DEFINITION 5.2. An $L_\infty$-algebra is equivalent to the following data:

A graded vector space $W = \oplus W_i$.

The **free graded commutative coalgebra** generated by $sW$, which is $W$ with the grading shifted up by 1, denoted

$$\Lambda sW = \bigoplus \Lambda^n sW.$$

A linear map

$$D = d_0 + d_1 + d_2 + \cdots : \Lambda sW \to \Lambda sW$$

where each $d_i$ is a **co-**derivation which lowers $n$ by $i$ such that $D^2 = 0$. We say that $D$ is a coderivation to summarize the several conditions:

$$d_i(x_1 \wedge \cdots \wedge x_n) = \Sigma \pm d_i(x_{j_1} \wedge \cdots \wedge x_{j_{i+1}}) \wedge x_{j_{i+2}} \wedge \cdots \wedge x_{j_n}$$

where the sum is over all *unshuffles* of $\{1, \dots, n\}$, that is, all permutations that keep each of the subsets $1, \dots, i$ and $i+1, \dots, n$ in the same relative order. Thus, $d_i$ is determined by $d_i|_{W^{\otimes i+1}} : W^{\otimes i+1} \to W$.

In accordance with the physics conventions, the $V$ of Definition 5.1 is the $sW$ of Definition 5.2.

Even ordinary Lie algebras came late to being defined operadically [**25**], although an appropriate operad was already there implicitly in the work of Fred Cohen [**16**].

DEFINITION 5.3. The Lie operad $\mathcal{L} = \{\mathcal{L}(n)\}$ is defined by $\{\mathcal{L}(n) := H_{n-1}(F(\mathbb{R}^2, n))$ for $n \geq 1\}$, where $F(\mathbb{R}^2, n)$ denotes the configuration space of ordered $n$-tuples of distinct points in $\mathbb{R}^2$.

The operad structure is best seen via the deformation retractions $\mathcal{D}(n) \to F(\mathbb{R}^2, n)$, where $\mathcal{D}$ is the little disks operad.

REMARK 5.2. The $L_\infty$-operad is more subtle. The first description is due to Hinich and Schechtman [**25**]. According to their theorem, $L_\infty$-algebras can be described as algebras over a certain linear operad built from trees, which can be identified with an operad constructed from one row of the $E^1$ term of the spectral sequence derived from the filtration by strata of the real compactification of the moduli spaces of punctured Riemann spheres [**31**]. (An operad structure on the stratification spectral sequence of the DKM-compactification was first observed by Beilinson and Ginzburg [**11**].)

A beautiful extension of these ideas can be found in Ginzburg and Kapranov [**22**].

**5.2. BRST cohomology and $L_\infty$-algebras.** BRS refers to Becchi, Rouet and Stora, who in 1975 [**10**] called attention to the "so-called Slavnov identities which express an invariance of the Fadde'ev-Popov Lagrangian". The T refers to Tyutin who, at about the same time [**51**], had a preprint on the same subject - the symmetry of gauge transformations.

The acronym BRST has come to be applied in mathematical physics very widely; it seems most justified in the following context. From the point of view of differential homological algebra, BRST cohomology is a generalization of Lie algebra cohomology (with coefficients in a module); the coboundary operator typically written in terms of a basis contains a term of the same form as that of Chevalley-Eilenberg. The generalization involves a differential graded vector space $L$ with a bracket and an action of $L$ on a differential graded vector space $M$ (usually a Koszul-Tate resolution), but both the Jacobi identity and the representation property are no longer satisfied precisely but only up to homotopy, in the strong sense that the BRST differential corresponds to an $L_\infty$-structure on $L \oplus M$.

In the Lagrangian formalism, this $L_\infty$-algebra appears as a deformation of an abelian graded Lie algebra. The physicists' 'master equation' [**9**] is precisely the integrability equation [**20**] of formal algebraic deformation theory [**8**].

## 6. Gerstenhaber algebras and BV-algebras

The full homologies $H_*(F(\mathbb{R}^2, n))$ also form an operad in the category of graded vector spaces. Since $H_*(F(\mathbb{R}^2, 2))$ is 0 except for $H_0 \approx H_1 \approx k$, we have two basic operations: a graded commutative product (denoted $a \otimes b \mapsto ab$) and a Whitehead bracket $[\ ,\ ]$ of degree 1 (corresponding to a graded Lie bracket after suspension). The defining relation between the two operations, a Leibniz identity with suitable signs:

$$[a, bc] = [a, b]c + (-1)^{|a+1||b|} b[a, c]$$

is that of a Gerstenhaber algebra [**19**]. Gerstenhaber defined his bracket in the context of Hochschild cohomology of an algebra with coefficients in itself precisely by constructing the $\circ_i$-operations of a comp algebra structure on Hochschild's cochain complex. However, various of the tri-linear relations on the cohomology held only up to homotopy on the cochain level. A similar story holds in certain phyically inspired examples [**37**], [**33**]. Which set of homotopies are to be the basis for a definition of "homotopy Gerstenhaber algebra" should be dictated by applications. Those we now are beginning to see in physics and physically inspired mathematics lead to the various operads discussed in [**33**], especially the one denoted $G_\infty$.

Still more complicated is what is now known as a BV-algebra (Batalin-Vilkovisky) [**9**], again motivated by structures inspired by mathematical physics, which can simplistically be described as a Gerstenhaber algebra with an additional operator $\Delta$ which is a derivation with respect to the bracket such that $\Delta^2 = 0$ but

$$[a, b] = \Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b);$$

that is, the failure of $\Delta$ to be a derivation of the graded commutative product is given by the bracket. Since $\Delta$ is a derivation of the bracket, it can be described as a 'differential operator of order 2'. This has motivated Akman [**1**] to study 'differential operators of order $r$' in the context of algebras $A$ without any assumption of commutativity - or even of associativity! (Her definition agrees with Koszul's [**35**] if the algebra is commutative, but differs from that of Grothendieck [**23**] in the most general case.) For any linear map $\Delta : A \to A$, Akman defines inductively a sequence of obstructions $\Phi^{r+1}$ which are $(r+1)$-linear forms with values in $A$. The operator $\Delta$ is **differential of order** $\le r$ if $\Phi^{r+1} = 0$. Remarkably, if $A$ is (graded) commutative and $\Delta^2 = 0$, then with $\Phi^0 = \Delta$, the sequence of $\Phi^r$'s forms an $L_\infty$-algebra. If $A$ is not (graded) commutative but $\Delta^2 = 0$, Akman's relations among the $\Phi^r$'s provides a good definition of a (strong homotopy) *Leibniz*$_\infty$-algebra.

## 7. Operads and Conformal Field Theories

As mentioned in the introduction, operads revealed themselves in the physical context of String Field Theories (SFTs) and Vertex Operator Algebras (VOAs), both of which in turn draw on Conformal Field Theories (CFTs).

**7.1. The moduli space operad of Riemann spheres with coordinatized punctures.** Instead of creating an operad from the moduli spaces of punctured Riemann surfaces by compactifying, one can also succeed by decorating the punctures with local coordinates. We can 'sew' two Riemann surfaces together unambiguously (up to the modular equivalence) if we have suitable local coordinates at the punctures. The term 'sewing' occurred in the physics literature for some time before Huang, in the contex of vertex opertor algebras, gave it an operadic formulation [**26**].

Let $\mathcal{P}_n$ be the moduli space of nondegenerate Riemann spheres $\Sigma$ with $n$ *labelled* punctures and non-overlapping holomorphic disks at each puncture (holomorphic embeddings of the standard disk $|z| < 1$ to $\Sigma$ centered at the puncture). The spaces $\mathcal{P}_{n+1}$, $n \geq 1$, form an operad under sewing Riemann spheres at punctures (cutting out the disks $|z| \leq r$ and $|w| \leq r$ for some $r = 1 - \epsilon$ at sewn punctures and identifying the annuli $r < |z| < 1/r$ and $r < |w| < 1/r$ via $w = 1/z$). The symmetric group interchanges punctures along with the holomorphic disks, as usual.

The essence of a CFT can now be described as follows [**32**]:

Consider the Virasoro algebra Vir, which is the algebra of complex-valued vector fields on the circle in this text. Vir is generated by the elements $L_m = z^{m+1}\partial/\partial z$, $m \in \mathbb{Z}$, with the commutators given by the formula $[L_m, L_n] = (n - m)L_{m+n}$. By $V$ we will denote the complexification of this algebra, $V := \mathrm{Vir} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Vir} \oplus \overline{\mathrm{Vir}}$.

A *CFT* (*at the tree level*) consists of the following *data*:

1. A topological vector space $\mathcal{H}$ (a *state space*).
2. An action $T : V \otimes \mathcal{H} \to \mathcal{H}$ of the complexified Virasoro algebra $V$ on $\mathcal{H}$.
3. A vector $|\Sigma\rangle \in \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H})$ for each $\Sigma \in \mathcal{P}_{n+1}$ depending smoothly on $\Sigma$.

These data must satisfy the following compatibility *axioms*:

4. $T(\mathbf{v})|\Sigma\rangle = |\delta(\mathbf{v})\Sigma\rangle$, where $\mathbf{v} = (v_1, \dots, v_{n+1}) \in V$ and $\delta$ is the natural action of $V^{n+1}$ on $\mathcal{P}_{n+1}$ by infinitesimal reparameterizations at punctures. In particular, $T(\mathbf{v})|\Sigma\rangle = T(\overline{\mathbf{v}})|\Sigma\rangle = 0$, whenever $\mathbf{v}$ can be extended to a holomorphic vector field on $\Sigma$ outside of the disks.
5. The correspondence $\Sigma \mapsto |\Sigma\rangle$ defines the structure of an algebra over the operad $\mathcal{P}_{n+1}$ on the space of states $\mathcal{H}$.

No physicist would express the last axiom in those terms, and certainly not the more succinct: a CFT is an algebra over the operad $\mathcal{P}_{n+1}$, implying an action of $V^{n+1}$ on the states $|\Sigma\rangle$.

**7.2. $L_\infty$-algebras and Closed String Field Theory.** A CSFT is built on a special kind of CFT. A *string background* (*at the tree level*) is a CFT based on a vector space $\mathcal{H}$ with the following additional *data*:

1. A $\mathbb{Z}$-grading $\mathcal{H} = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}_i$ on the state space.
2. An action of the Clifford algebra $C(V \oplus V^*)$, which is denoted usually by $b : V \otimes \mathcal{H} \to \mathcal{H}$ and $c : V^* \otimes \mathcal{H} \to \mathcal{H}$ for generators of the Clifford algebra, the degree of $b$ is $-1$, and the degree of $c$ is 1.
3. A differential $Q : \mathcal{H} \to \mathcal{H}$, $Q^2 = 0$, of degree 1, called a *BRST operator*, such that
4. $Qb + bQ = T$.

The graded space $\mathcal{H}$ with the operator $Q$ is called a *BRST complex*. For $\psi \in \mathcal{H}_i$, the degree $\operatorname{gh} \psi := i$ is called the *ghost number*. The BRST complex here refers to that for the Virasoro algebra.

One of the nicest implications of a string background is the construction of a morphism of complexes $\Omega_{n+1} : \operatorname{Hom}(\mathcal{H}, \mathcal{H}^{\otimes n}) \to \Omega^\bullet(\mathcal{P}_{n+1})$, $n \geq 1$, from the complex of linear mappings between tensor powers of the BRST complex $\mathcal{H}$ to the de Rham complex of the space $\mathcal{P}_{n+1}$.

The physicists' notion of a CSFT can be rephrased in our terms.

Definition 7.1. A CSFT (Closed String Field Theory) is a string background together with a morphism of operads $\underline{\mathcal{M}} \to \mathcal{P}$.

A CSFT defines the structure of a homotopy Lie algebra on the space $\mathcal{H}_{\mathrm{rel}}$ of relative states. The brackets defining this structure are given by the formula:

$$[\cdot, \dots, \cdot] = \int_{\underline{\mathcal{M}}_{n+1}} \omega_{n+1} \in \operatorname{Hom}((\mathcal{H}_{\mathrm{rel}})^n, \mathcal{H}_{\mathrm{rel}})$$

where $\underline{\mathcal{M}}_{n+1}$ is yet another compactified and decorated moduli space of dimension $2n-4$. The form $\omega_{n+1}$ is related to $\Omega_{n+1}$ by a pull-back.

The homotopy Lie algebra structure in CSFT was first constructed by Zwiebach [**54**], but it was in [**31**] that the moduli spaces $\underline{\mathcal{M}}_{n+1}$ were defined and the homotopy Lie structure was explained as coming just from geometry of the $\underline{\mathcal{M}}_{n+1}$'s.

## 8. Homotopy commutativity and $C_\infty$-algebras

One issue noticeably missing in the above discussion but prominent in the earliest use of operads is that of homotopy commutativity. From the point of view of Koszul duality [**22**], in which Lie algebras and commutative algebras are dual, the concept Koszul dual to an $L_\infty$-algebra is that of a $C_\infty$-algebra:

DEFINITION 8.1. A $C_\infty$-algebra is equivalent to the following data:

A graded vector space $W = \oplus W_i$.

The **free graded Lie coalgebra** generated by $sW$, which is $W$ with the grading shifted up by 1, denoted

$$\mathcal{L}sW = \bigoplus \mathcal{L}^n sW.$$

A linear map

$$D = d_0 + d_1 + d_2 + \cdots : \mathcal{L}sW \to \mathcal{L}sW$$

where each $d_i$ is a **co-**derivation which lowers $n$ by $i$ such that $D^2 = 0$. Thus, $d_i$ is determined by $d_i$ taking $i$-fold brackets to elements of $W$.

Notice that a $C_\infty$-algebra is an $A_\infty$-algebra with strict commutativity of the bilinear product, not homotopy commutativity, and suitable symmetry of the higher order products. In the context of (strong) homotopy algebras as given by multilinear operations on $W$, the simplest definition of a $C_\infty$-algebra first appeared in work of Kadeishvili [**29, 28**] and then in that of Smirnov [**47**] (both of whom called them commutative $A_\infty$-algebras). Such algebras appeared also in [**39**] under the name of 'balanced $A_\infty$-algebras'.

DEFINITION 8.2. A $C_\infty$-*algebra* is an $A_\infty$-algebra $(A, \{m_n\})$ such that each map $m_n : A^{\otimes n} \to A$ is a Harrison cochain, i.e. $m_n$ vanishes on the sum of all $(p, q)$-shuffles for $p + q = n$, the sign of the shuffle coming from the grading of $A$ shifted by 1.

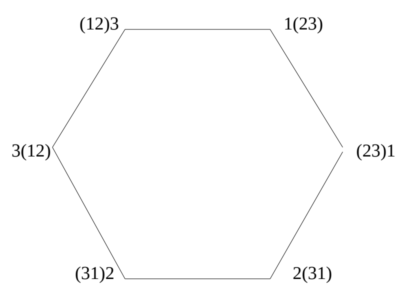
In characteristic 0, when confronted with a bilinear product which is only homotopy commutative, we can always symmetrize to obtain strict (graded) commutativity. The price we pay if the original bilinear product were associative is that the symmetrized one no longer is. It therefore is in organizing the higher homotopies that we may be led to choose one alternative over the other. This issue also arises for $L_\infty$-algebras and even more dramatically for homotopy Gerstenhaber or BV-algebras [**37, 33**].

The relevance of $C_\infty$-algebras to moduli spaces and physics ('filtered topological gravity') is pointed out in [**32**] where we consider the operad of Deligne-Knudsen-Mumford compactifications of the moduli spaces of punctured Riemann spheres and the corresponding operad $\mathcal{S}$ of singular chains.

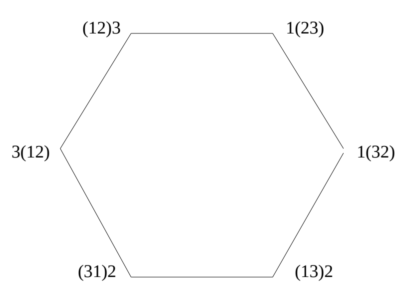
Let $V$ be an algebra over $\mathcal{S}$ such that the structure morphism $\mu : \mathcal{S}(n) \to \mathrm{Hom}(V^{\otimes n}, V)$ vanishes on all elements in $\mathcal{S}(n)$ of chain degree $p > n-2$; then $V$ has the structure of a $C_\infty$-algebra.

If we choose not to symmetrize but have strict associativity, then the permutahedra $P_n$ [**44, 30**] are relevant. If we have both homotopy commutativity and homotopy associativity, then we confront the permutassociahedra $KP_n$ [**30**] which are relevant to Mac Lane's coherence conditions [**38**] or the full panoply of $E_\infty$-operads as they were originally intended.

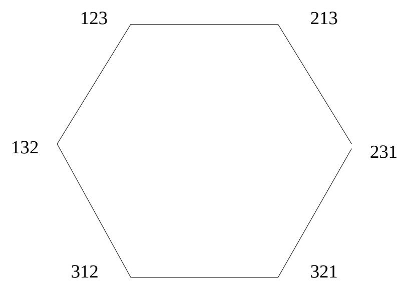
Notice that we have *different* hexagons in these various settings, most easily distinguished by noting the vertex labellings.




$W_3$

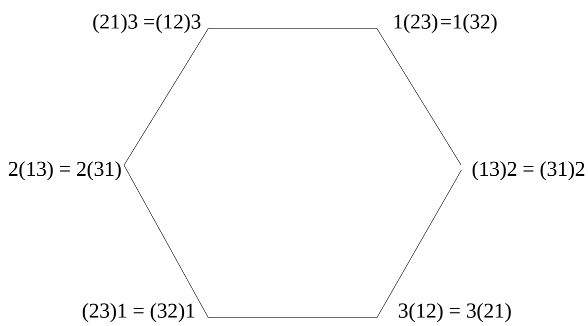



$KP_3$ Mac Lane's hexagon

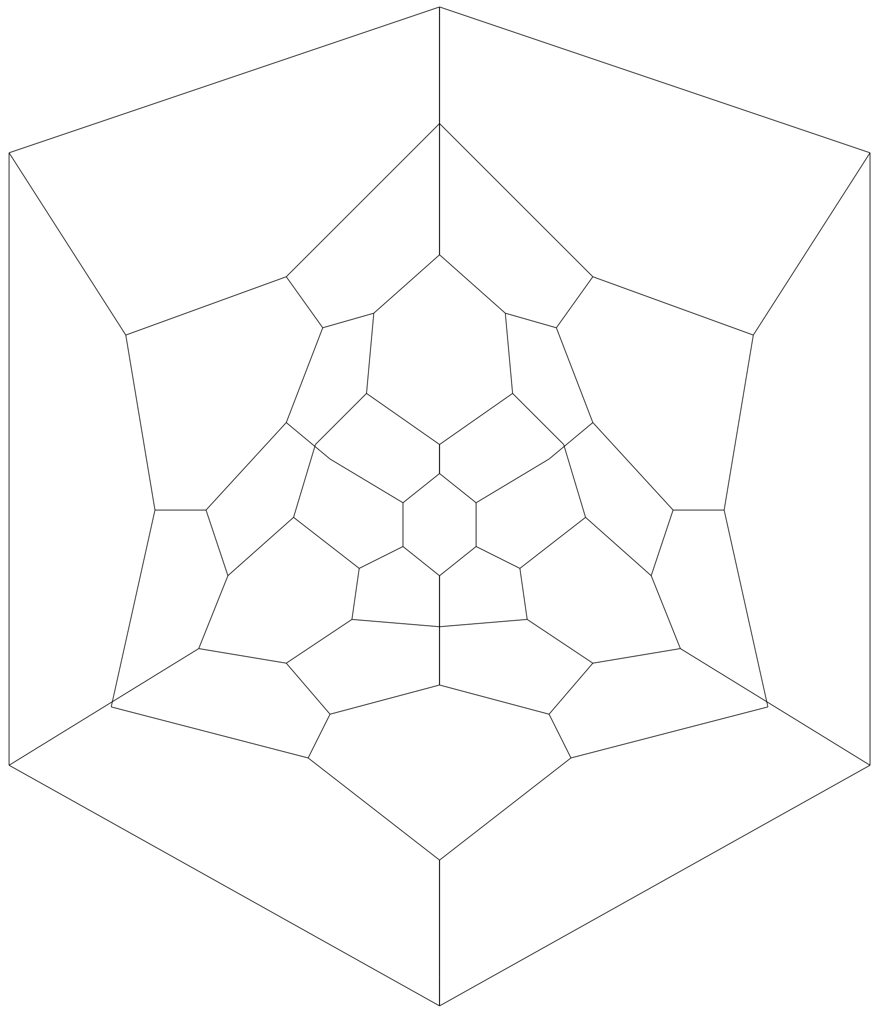



$P_3$

Finally?, if we have strict commutativity and homotopy associativity, we can consider the hexagon




which together with the pentagon $K_4$ can give rise to the 'alternate universe soccer ball', the vertices being given by all meaningful bracketings of 4 variables subject to the commutativity $xy = yx$.



## 9. Appendix A: Knot invariants and Cyclohedra

It all began with Gauss in 1833. He defined the linking number of two disjoint closed or infinite curves:

$$insertGauss'original$$

Of *Geometria Situs*, that Leibnitz guessed and of which only a pair of geometers (Euler and Vandermonde) were privilieged to have had a weak sight, we know not much more that nothing after a century and a half.

A major task from the boundary of *Geometria Situs* and *Geometria Magnitudinus* would be to count the linking number of two closed or infinite curves.

Let the coordinates of an arbitrary point on the first curve be $x, y, z$; on the second $x', y', z'$ and

$$\int\int \frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z-z')(dxdy'-dydx')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}} = V$$

then the integral extended over both curves

$$= 4\pi m$$

and $m$ is the linking number.

The value is symmetric, i.e. it remains the same, if the two curves are interchanged. 1833, Jan. 22.

Today with vector and differential form notation, we can write this more succinctly as follows:

$$\begin{aligned} \phi : S^1 \times S^1 &\longrightarrow \mathbb{R}^3 - 0 \quad \simeq \quad S^2 \\ (x,y) &\longmapsto \frac{K_1(x) - K_2(y)}{|K_1(x) - K_2(y)|} \\ Link(K_1, K_2) &= 1/4\pi \int_{S^1\times S^1} \phi^*(d\,\mathrm{vol}_{S^2}). \end{aligned}$$

At a lecture at MSRI, Bott presented some of the work which became his paper with Taubes: 'On the self-linking of knots' [**14**]. Their purpose was 'to present a "topological" account of the self-linking invariant of a knot $K \subset \mathbb{R}^3$, discovered independently by Guadagnini, Martellini and Mintchev [**24**] and Bar-Natan [**7, 6**]'. They were highly motivated by a generalization due to Kontsevich [**34**] which can be regarded as being in the Gaussian tradition, once we realize Gauss' integral is over the configuration space of pairs of distinct points, one on each of the two curves.

Bott and Taubes use the notation $C(n, \mathbb{R}^3)$ for configurations of $n$ ordered points in $\mathbb{R}^3$, also known as $F(\mathbb{R}^3, n)$ above, and $\overline{C}(n, \mathbb{R}^3)$ for the compactification. We will stay with the notation $F$ and $\overline{F}$ as in the body of this paper.

Recall the map $\phi$ used in defining the Gauss linking number, or rather, the retraction $\mathbb{R}^3 - 0 \;\to\; S^2$. Similarly, for $F(n, \mathbb{R}^3)$ we have several maps $\phi_{ij} : F(n, \mathbb{R}^3) \to S^2$ given by

$$(x_1, \ldots, x_n) \mapsto \frac{x_i - x_j}{|x_i - x_j|} \text{ for } \; i \neq j.$$

(Bott-Taubes) The maps $\phi_{ij}$ extend smoothly to the compactifications $\overline{F}(n, \mathbb{R}^3)$. We denote by $\theta_{ij}$ the pullback under $\phi_{ij}$ of the (normalized) volume form on $S^2$. The forms $\theta_{ij}$ generate $H^*(F(n, \mathbb{R}^3))$ [**16, 3**], though their relation to $H^*(\overline{C}(n, S^3))$ is less clear.

Since a knot $K$ is an imbedding of $S^1$ in $\mathbb{R}^3$, we have an induced map

$$\overline{F}(n, K) : \overline{F}(n, S^1) \to \overline{F}(n, \mathbb{R}^3)$$

and can further pull back the forms $\theta_{ij}$ to $\overline{F}(n, S^1)$. Recall that $\overline{F}(n, S^1)$ can be identified with $S^1 \times W_n$ as we have reindexed the $W_n$ by the number of points, rather than its dimension as a topological space. To get a numerical potential knot invariant, we can integrate a polynomial in the $\theta_{ij}$ over $\overline{F}(n, S^1)$ if the polynomial has degree equal to one half the dimension of $\overline{F}(n, S^1)$. (It is for purposes of such integration that Kontsevich introduced the compactifications.)

To verify if the resulting number is a knot invariant, we must consider its behavior under an isotopy of the knot. For this purpose, let $\mathcal{K}$ denote the space of knots, i.e. of smooth embeddings of $S^1$ in $\mathbb{R}^3$ and consider the map

$$\mathcal{K} \times \overline{C}(n, S^1) \to \overline{C}(n, \mathbb{R}^3)$$
$$(K, x) \mapsto \overline{C}(n, K)(x).$$

Now we can fibre integrate a polynomial in the (pullbacks of the) $\theta_{ij}$ over $\overline{F}(n, S^1)$ to obtain a numerical function on $\mathcal{K}$. It will be a knot invariant if it is constant on each path component of $\mathcal{K}$, i.e. if it is locally constant on $\mathcal{K}$ or, equivalently, has exterior derivative equal to zero. Computing the exterior derivative, we see that 'boundary terms' intrude as integrals over the facets of $W_n$. To obtain a knot invariant, the trick is to add counter terms involving configurations of $n+t$ points of which $n$ are on the knot and the remaining $t$ anywhere else in $\mathbb{R}^3$. The first example, $v_2$, of such an invariant occurred in the physics literature in a paper by Guadagnini, Martellini and Mintchev [**24**] and was worked out independently by Bar-Natan [**6**].

To simplify the notation in the integrals, we denote $\overline{F}(n, \mathbb{R}^3)$ by $F_n$ and the compactification of configurations of $n+t$ points with $n$ on the knot by $F_{n,t}$. We can then write this invariant as

$$v_2 = 1/4 \int_{F_4} \theta_{13}\theta_{24} - 1/3 \int_{F_{3,1}} \theta_{14}\theta_{24}\theta_{34}.$$

The next example appears in a paper of Alvarez and Labastida [**2**]; as transcribed with some correction of signs by Dylan Thurston [**50**], $v_3$ can be written as:

$$1/2 \int_{F_6} \theta_{14}\theta_{26}\theta_{35} - 1/3 \int_{F_6} \theta_{14}\theta_{25}\theta_{36} + \int_{F_{5,1}} \theta_{16}\theta_{25}\theta_{36}\theta_{46} + 1/2 \int_{F_{4,2}} \theta_{15}\theta_{25}\theta_{56}\theta_{36}\theta_{46}.$$

Remarkably, in almost all cases, it can be shown that only the 'principal' faces of the form $W_{n-1}$ require the addition of counterterms. At the present writing, a few troublesome and very special cases remain to be verified and a conceptual approach to this aspect of the need for counterterms is still missing.

## 10. Appendix B: Associahedra and Cyclohedra as truncated simplices (with Steven Shnider)

We will construct a convex polyhedron $L_n$ isomorphic to the associahedron $K_n$ by truncation of a simplex and then a simple modification of the construction will produce a realization of the cyclohedron $W_n$.

We use the following realization of the simplex as a subset of $\mathbf{R}^{n-1}$,

$$\Delta^{n-2} = \{t = (t_1, \dots, t_{n-1}) | t_i \geq 0, \sum_{k=1}^{n-1} t_i = c_{n-1}\},$$

and denote the vertices of $\Delta^{n-2}$ by $e_k$ for $1 \leq k \leq n-1$. In order to keep track of the truncation procedure, it will be helpful to let the value of the constant $c_k$ depend on $k$. The truncation procedure will be by hyperplanes parallel to some, but not all, faces of $\Delta^{n-2}$. The relevant ones will correspond to sets of vertices indexed by intervals $I = (i, \dots, j)$ of natural numbers, so we consider the class of functions from intervals of natural numbers to the positive reals satisfying the following conditions:

1. $c(I) > 0$ for all intervals $I$
2. $c(I_1) + c(I_2) < c(I_1 \cup I_2)$ if $I_1 \cup I_2$ properly contains both $I_1$ and $I_2$.

We call these functions "suitable". For example, let $c(I) = 3^{\#I}$. Having chosen such a function, we set $c_k = c((1, \cdots, k))$.

For any interval $I \subset (1, \dots, n-1)$, the face $\Delta_I$ of $\Delta^{n-2}$ is the one that *excludes* the vertices $e_k$ for $k \in I$ and thus is defined by

$$\Delta_I = \{t \in \Delta^{n-2} | \sum_{k \in I} t_k = 0\}.$$

For any function $c(I)$ satisfying the conditions above we define a family of truncating hyperplanes,

$$P_I^c = \{t | \sum_{k \in I} t_k = c(I)\}.$$

Define an $n-2$ dimensional convex polytope in $\mathbf{R}^{n-1}$ by

$$L_n^c = \{t | \sum_{k=1}^{n-1} t_k = c_{n-1}, \quad \sum_{k \in I} t_k \geq c(I) \quad \text{for each} \quad I \subset (1, \dots, n-1)\}.$$

We shall prove For any suitable function $c$, there is an isomorphism of cell complexes between $L_n^c$ and the cell complex $K_n$, where the $k$ cells of $L_n^c$ are its $k$-dimensional faces.

The proof is based on a lemma, which uses the concept of compatible intervals of indices. Two intervals of natural numbers, $I$ and $J$, will be called *compatible* if $I \cup J$ is not an interval properly containing both $I$ and $J$, that is, either

1. $J \subset I$, or
2. $I \subset J$, or
3. $I \cup J$ is not an interval.

For the purpose of stating the lemma, a single interval will be said to be compatible with itself.

LEMMA 10.1. *The polytope $L_n^c$ has nonempty interior in the $n-2$ dimensional hyperplane $\{t | \sum t_j = c_{n-1}\}$ for all suitable functions $c$. The intersection over $p$ hyperplanes, $L_n^c \cap \bigcap P_{I_a}^c$, defines a nonempty $(n-p-2)$-face if and only if the intervals $I_a$ are pairwise compatible.*

First we show that the intersection $P_I \cap P_J \cap L_c^n$ is empty if the intervals $I$ and $J$ are not compatible. Indeed, the conditions on the function $c$ imply that when $I \cup J$ is an interval properly containing both $I$ and $J$, then for $t \in P_I^c \cap P_J^c$ and $t_i \geq 0$, we have

$$\sum_{i \in I \cup J} t_i \leq \sum_{i \in I} t_i + \sum_{i \in J} t_i = c(I) + c(J) < c(I \cup J), \tag{10.1}$$

violating the inequality for the interval $I \cup J$ which appears in the definition of $L_n$.

The rest of the proof of the lemma is by induction on $n$ using the following reduction procedure. Suppose the values of $t_j$ for $k \leq j \leq k+s-1$ have been fixed so that the sum is $c((k, \dots, k+s-1))$, for some pair $(k, s)$ where $1 \leq k \leq n-1$ and $1 \leq s \leq n-k$. Define an associated modification, $c_{k,s}$, of the function $c$. For any interval $I = (i, \cdots, j)$, let $I_{+s}$ be the interval $(i, \dots, j+s)$. Define $c_{k,s}$ by

1. $c_{k,s}(I) = c(I_{+s}) - c((k, \cdots, k+s-1))$ if $k \in I$,
2. $c_{k,s}(I) = c(I)$ if $k \notin I$.

One verifies immediately that $c_{k,s}$ is suitable if $c$ was. Define new variables $t'_j$,

1. $t'_j = t_j$ for $j < k$,
2. $t'_j = t_{j+s}$ for $k \leq j \leq n-s-1$.

Once the values of $t_j$ in the given interval $(k, \dots, k+s-1)$ are fixed the inequalities for the the remaining $t_j$ outside this interval are the same as for the polytope $L_{n-s}^{c_{k,s}}$ in the $n-s-1$ space with variables $t'_j$.

To prove that $L_n^c$ has nonempty interior in the $n-2$ hyperplane, we first show that $L_c^n$ is nonempty by an easy induction, using the construction just given, which shows that the intersection of the affine hyperplane $t_1 = c((1))$ with the polytope $L_n^c$ is a polytope $L_{n-1}^{c'}$. Beginning with $L_2^c$, which is a point for any $c$, we proceed inductively to show that $L_n^c$ is nonempty. Next for a given suitable function and a fixed $n$, there exists an $\epsilon > 0$ such that setting $c'_{n-1} = c_{n-1}$ and $c'(I) = (1+\epsilon)c(I)$ for any proper subinterval of $(1,\ldots,n-1)$ defines a suitable function on the subintervals of $(1,\ldots,n-1)$. A point of $L_n^{c'}$ is an interior point of $L_n^c$ as a set in $n-2$ space.

Finally, we use induction on $n$ to prove that the intersection $L_n^c \cap \bigcap P_{I_a}^c$ is a nonempty $(n-p-2)$-dimensional set if the $p$ intervals, $I_a$ are compatible. Compatibility implies that the system of linear equations defining $\cap P_{I_a}$ has rank $p$, so if the intersection is not empty, it has dimension $n-p-2$. Label the intervals so that $I_1$ does not contain any other interval $I_a$. Assume $I_1 = (k,\ldots,k+s-1)$. Since $L_{s+1}^c$ is nonempty, there is a point $(t_k,\ldots,t_{k+s-1})$ satisfying all the required conditions which involve only these coordinates. Consider the polytope $L_{n-s}^{c_{k,s}}$ in the coordinates $t'_j$ defined above. The intervals of indices in the $t'_j$ coordinates are compatible and so by the induction assumption, the intersection of the corresponding hyperplanes $P'_{I'_a}$ for $a > 1$ with the polytope $L_{n-s}^{c_{k,s}}$ is nonempty. To complete the proof we must show that returning to the original coordinates, $(t_1,\ldots,t_k,\ldots,t_{k+s-1},\ldots,t_{n-1})$ is in the required intersection. We need only verify the inequalities in the definition of $L_n^c$ which involve intervals which are not compatible with $I_1$, namely those intervals $J$ which intersect $I_1$ but do not contain $I_1$ and such that $I_1 \cup J$ properly contains $I_1$, since all others involve either the full sum $t_k + \ldots + t_{k+s-1}$ as a summand or involve only the coordinates $\{t_k,\ldots,t_{k+s-1}\}$. Let $J' = I_1 \cap J$ and $J'' = J - J'$, then

$$\sum_{j\in J} t_j = \sum_{j\in J'} t_j + \sum_{j\in J''} t_j \geq c(J') + c(I_1 \cup J) - c(I_1) \geq c(J).$$

This concludes the proof of the lemma.

In $K_n$ the $p$-dimensional cells are labelled by bracketings of $n$ $\bullet$'s with $n-2-p$ pairs of parentheses and two cells intersect if and only if the two bracketing labels are consistent with a single bracketing, that is, there is a single big bracketing such that each of the given ones consists of a subset of the parentheses in the big bracketing.

We can use the same labelling for the faces of $L_n^c$. For $I = (k,\ldots,k+s-1)$, let $b(I)$ be the bracketing which is represented by a row of $n$ $\bullet$'s with parentheses around the $s+1$ $\bullet$'s in positions $k$ to $k+s$. For the singleton intervals $(k)$, $b((k))$ is a sequence of $n$ $\bullet$'s with parentheses around the $k^{th}$ and $k+1^{st}$. With this convention, for a set of $p$ compatible intervals $\{I_1,\ldots,I_p\}$ the corresponding bracketings $b(I_a)$ are consistent with a single bracketing with $p$ parentheses, $b(I_1,\ldots,I_p)$, which will label the intersection. Note that if $I_j$ and $I_k$ are compatible, then the bracketings $b(I_j)$ and $b(I_k)$ are either nested or disjoint. Adjacency for the intervals $I_j, I_k$ would imply that the bracketings in the $\bullet$'s overlap. This gives the following lemma, which finishes the proof of the proposition.

LEMMA 10.2. *The map $P_I \mapsto b(I)$ induces an incidence isomorphism between (the poset of) the cells of $L_n$ and those of $K_n$.*

To truncate the simplex $\Delta^{n-1}$ to realize $W_n$, we need only interpret the interval $I = (i, \ldots, j) \subset (1, \ldots, n-1)$ as consisting of integers *mod* $n-1$ and similarly for anything indexed by such an interval.

*Acknowledgments.* This work is partly a result of my collaboration with Kimura and Voronov. It has also benefited from interactions with F. Akman, G. Barnich, R. Bott, C. Taubes, M. Kontsevich, M. Markl, S. Shnider and D. Thurston.

## References

1. F. Akman, *On some generalizations of Batalin-Vilkovisky algebras*, Preprint, Cornell University, 1995, to appear in JPAA q-alg/9506027.
2. M. Alvarez and J.M.F. Labastida, *Vassiliev invariants for torus knots*, Tech. report, CTP-MIT, 1995, to appear in Journal of Knot Theory and its Ramifications, q-alg/9506009.
3. V. I. Arnold, *The cohomology ring of the colored braid group*, Mat. Zametki (1969), 227–231.
4. S. Axelrod and I. M. Singer, *Chern-Simons perturbation theory II*, J. Diff. Geom. **39** (1994), 173–213, hep-th/9304087.
5. D. Balavoine, *Deformation of algebras over a quadratic operad*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?
6. D. Bar-Natan, *On Vassiliev knot invariants*, Topology **34 No. 2** (1995), 423 – 472.
7. ______, *Perturbative aspects of the Chern-Simons topological quantum field theory*, J. Knot Theory and its Ram. **4** (1995), 503–548, a mildly updated version of his Princeton University Ph.D. thesis, 1991.
8. G. Barnich and M. Henneaux, *Consistent couplings between fields with a guage freedom and deformations of the master equation*, Phys. Lett. **B 311** (1993), 123–129.
9. I.A. Batalin and G.S. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. **102 B** (1981), 27–31.
10. C. Becchi, A. Rouet, and R. Stora, *Renormalization of the abelian Higgs-Kibble model*, Commun. Math. Phys. **42** (1975), 127–162.
11. A. Beilinson and V. Ginzburg, *Infinitesimal structure of moduli spaces of G-bundles*, Internat. Math. Research Notices **4** (1992), 63–74.
12. J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., vol. 347, Springer-Verlag, 1973.

13. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts*, Annals of Math. **57** (1953), 115–207.
14. R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys. **35** (1994), 5247–5287.
15. G. Calugareanu, *L'integral de Gauss et l'analyse de noueds tridimensionnels*, Rev. Math. Pures Appl **4** (1959), 5–20.
16. F. R. Cohen, *Artin's braid groups, classical homotopy theory and sundry other curiosities*, Contemp. Math. **78** (1988), 167–206.
17. V.G. Drinfel'd, *Quasi-Hopf algebras*, Leningrad Math J. **1** (1990), 1419–1457.
18. W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. Math. **139** (1994), 183–225.
19. M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. **78** (1962), 267–288.
20. M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. **79** (1964), 59–103.
21. E. Getzler and J.D.S. Jones, *n-algebras and Batalin-Vilkovisky algebras*, preprint, 1993.
22. V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), 203–272.
23. A. Grothendieck, *Elements de Geometrie Algebrique IV, Études Locals des Schemas et des Morphismes de Schemas*, Pub. Math. IHES **32** (1967), Proposition 16.8.8 on p.42.
24. E. Guadagnini, M. Martellini, and M. Mintchev, *Wilson lines in Chern-Simons theory and link invariants*, Nucl. Phys. **B330** (1990), 575–607.
25. V. Hinich and V. Schechtman, *Homotopy Lie algebras*, Adv. Studies Sov. Math. **16** (1993), 1–18.
26. Y.-Z. Huang, *Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras*, Comm. Math. Phys. (1994), 105–144, hep-th/9306021.
27. Stasheff J.D., *Differential graded lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Proceedings of the Workshop on Quantum Groups, Deformation Theory, and Representation Theory, Euler International Mathematical Institute, Leningrad, October 1990, vol. 1510, 1992, pp. 120–137.
28. T. Kadeishvili, *The category of differential coalgebras and the category of $A(\infty)$-algebras*, Proc. Tbilisi Math. Inst. **77** (1985), 50–70.
29. ———, *$A(\infty)$-algebra structure in cohomology and the rational homotopy type*, preprint 37, Forschungsschwerpunkt Geometrie, Universität Heidelberg, Mathematisches Institut, 1988.
30. M. M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure and Appl. Alg. **85** (1993), 119–142.
31. T. Kimura, J. Stasheff, and A. A. Voronov, *On operad structures of moduli spaces and string theory*, Commun. Math. Phys. **171** (1995), 1–25, hep-th/9307114.
32. T. Kimura, J. Stasheff, and A. A. Voronov, *Homology of moduli spaces of curves and commutative homotopy algebras*, The Gelfand Mathematics Seminars, 1993–1994 (J. Lepowsky and M. M. Smirnov, eds.), Birkhäuser, 1996.
33. T. Kimura, A.A. Voronov, and G. Zuckerman, *Homotopy Gerstenhaber algebras and topological field theory*, Operads: Proceedings of Renaissance Conferences

(J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?

34. M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992) (Basel), Progr. Math., vol. 120, Birkhäuser, 1994, pp. 97–121.
35. J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (1985), 257–271.
36. T. Lada and J.D. Stasheff, *Introduction to sh Lie algebras for physicists*, Intern'l J. Theor. Phys. **32** (1993), 1087–1103.
37. B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Commun. Math. Phys. **154** (1993), 613–646, hep-th/9211072.
38. S. Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), 28–46.
39. M. Markl, *A cohomology theory for $A(m)$-algebras and applications*, JPAA (1992), 141–175.
40. ———, *Models for operads*, Comm. in Algebra **24** (1996), 1471–1500.
41. W.S Massey, *Higher order linking numbers*, Conference on Algebraic Topology, 1968, pp. 174–205.
42. J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, 1972.
43. ———, *Definitions: Operads, algebras and modules*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?
44. R.J. Milgram, *Iterated loop spaces*, Annals of Math. **84** (1966), 386–403.
45. W.F. Pohl, *The self linking number of a closed space curve*, Jour. Math. and Mech. **17** (1968), 975–986.
46. M. Schlessinger and J. D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. of Pure and Appl. Alg. **38** (1985), 313–322.
47. V. A. Smirnov, *On the cochain complex of topological spaces*, Math. USSR Sbornik **43** (1992), 133–144.
48. J. Stasheff, *The pre-history of operads*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?
49. J. D. Stasheff, *On the homotopy associativity of H-spaces, I*, Trans. Amer. Math. Soc. **108** (1963), 275–292.
50. D. Thurston, *Integral expressions for the Vassiliev knot invariants*, Ph.D. thesis, Harvard Univ., 1995, senior thesis.
51. I. V. Tyutin, *Gauge invariance in field theory and statistical physics in operator formulation (in Russian)*, Tech. report, Lebedev Physics Inst., 1975.
52. E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in two-dimensional string theory*, Nucl. Phys. B **377** (1992), 55–112.
53. B. Zwiebach, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*, Nucl. Phys. B **390** (1993), 33–152.
54. ———, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*, Nucl. Phys. B **390** (1993), 33–152.

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
*E-mail address*: jds@@math.unc.edu